Compactness Approach for Stochastic Limits

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Outline

- Objective: Establish **limit of stochastic process**, which is later used in continuous mapping later. Or **weak convergence** of a sequence of probability measures.

- Main method: **Compactness approach**.
  To establish stochastic limits, we need to

  1. Prove that the sequence of probability measure is **relatively compact**.
  2. Prove all convergent subsequences convergence to the **same limit**.

Definitions

- **limit of a sequence of stochastic processes**
  The limit of a sequence of stochastic processes is linked to a sequence of probability measures.
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- **weak convergence** a.k.a **convergence in distribution**
  - We say that a sequence of probability measures $\{P_n : n \geq 1\}$ on metric space (S, m) *converges weakly* or *converges* to a probability measure P on (S, m), and we write $P_n \Rightarrow P$, if

    $$lim_{n\to\infty} \int_S f dP_n = \int_S f dP$$

    for all functions f in $C(S)$.
  - Equivalently, we have the Prohorov metric π and *weak convergence* is thus defined as if $\pi(P_n, P) \to 0$, then $P_n \Rightarrow P$.

Definitions

- **relatively compact**
  - If every sequence $\{P_n : n \geq 1\}$ in A has a subsequence $\{P_{n'} : n' \geq 1\}$ with $P_{n'} \Rightarrow P'$, where P' is necessarily in $\bar{A}$, then the set A is called **relatively compact**.
- **tightness**
  - A subset A of probability measures in $P(S)$ is said to be **tight** if, $\forall \epsilon$, there $\exists$ a **compact** subset $K(\epsilon)$ of (S, m), such that

$$P(K) > 1 - \epsilon, \text{for all } P \in A$$

  - Intuition: for a given collection of probability measures, none of them goes to infinity.
  - How is **tightness** related to **relatively compactness**?

Flow of idea

Primarily, we focus on proving a set being relatively compact, hence the word **compactness**. Furthermore, we don't usually directly prove compactness of a set A , instead we try to prove its equivalent property, **tightness**.

The equivalence is valid only under specific condition and is characterized in **Prohorov's Theorem**.

However, the second part, proving the limit being the same, or uniqueness of the limit, varies for different questions. It is therefore left out in here.

Equivalence between compactness and tightness

Theorem (Prohorov's Theorem)

Let (S, m) be a metric space. If a subset A in $P(S)$ is tight, then it is relatively compact. On the other hand, if the subset A is relatively compact the topological space S is Polish, then A is tight.

Note: for a space to be Polish, we need to establish separability and topological completeness.

Therefore, the specific condition mentioned previously is for the space S to be *Polish*. That is, in Polish space tightness is necessary and sufficient for relative compactness.

Tightness criterion for weak convergence

A direct result from Prohorov's Theorem is the following:

Corollary (Tightness criterion for weak convergence)

Let $\{P_n : n \geq 1\}$ be a sequence of probability measures on a metric space (S, m). If the sequence $\{P_n\}$ is tight and the limit of any convergent subsequence from $\{P_n\}$ must be P, then $P_n \Rightarrow P$.

Note that tightness establishes relatively compactness.

Criteria for convergence in distribution in $\mathcal{C}$

Turn the part of the statement in Corollary 11.6.1 that the limit of any convergent subsequence from $\{P_n\}$ must be P in terms of convergence of all the finite-dimensional distributions of X_n to those of X results the following corollary.

Corollary (Criteria for convergence in distribution in $\mathcal{C}$)

There is convergence $X_n \Rightarrow X$ in $\mathcal{C}$ iff the sequence $\{X_n : n \geq 1\}$ is tight and there is convergence of all the finite-dimensional distributions of X_n to those of X.

Tightness establish relatively compactness. Together with convergence of all the finite-dimensional distributions of X_n to those of X gives you weak convergence.

Characterize tightness in $\mathcal{C}$

To characterize tightness of a sequence $\{X_n : n \geq 1\}$, we introduce the following theorem.

Theorem (Tightness criterion for random elements of $\mathcal{C}$)

A sequence $\{X_n : n \geq 1\}$ of random elements of C is tight iff, $\forall \epsilon > 0$, $\exists$a constant c such that:

$$P(|X_n(0)| > 0) < \epsilon \ , \forall n \geq 1$$

And $\forall \epsilon > 0$ and $\eta > 0$, $\exists \delta > 0$ and n_0 such that,

$$P(v(X_n, \delta) \geq \epsilon) \leq \eta \ , \forall n \geq n_0$$

Note that

- The first inequality establishes $\{X_n(0)\}$ is tight
- $v(X_n, \delta)$ is defined as a *modulus of continuity*

Direct result from Tightness criterion for random elements of $\mathcal{C}$

We state the theorem first, which is another criteria for convergence in distribution in C.

Theorem (Criteria for convergence in distribution in $\mathcal{C}$)

There is convergence in distribution $X_n \Rightarrow X$ in $\mathcal{C}$ iff convergence of all the finite-dimensional distributions holds and two equations in Theorem 11.6.3 hold as well.

Notice that in here, the part that requires convergence of all the finite-dimensional distributions is identical to the half of the requirement in Theorem 11.6.2 and the rest of Theorem 11.6.4 establishes tightness of the sequence which is also identical to the other half in Theorem 11.6.2.

It is obvious that Theorem 11.6.4 elaborates Theorem 11.6.2 in terms of proving tightness.

Extension of criteria for tightness in $\mathcal{C}$

Replace the modulus inequality in Theorem 11.6.3, we extend criterion for tightness in $\mathcal{C}$.

Theorem (Theorem 11.6.5 Moment criterion for tightness in $\mathcal{C}$)

A sequence $\{X_n : n \geq 1\}$ of random elements of C is tight iff, $\{X_n(0)\}$ is tight and $\exists \gamma \geq 0$ and $\alpha > 1$ and a nondecreasing continuous function g on [0,1] such that,

$$E[|X_n(t) - X_n(s)|^\gamma] \leq |g(t) - g(s)|^\alpha, \ \forall 0 \leq s \leq t \leq 1$$

References