Introduction to Stochastic-Process Limits

Probability Seminar

Yunan Liu

Industrial and Systems Engineering
North Carolina State University

Why Do We Care?

Motivation

- Simple ***approximations*** for complicated stochastic processes

- Explain ***statistical regularity*** (appropriate scaling)

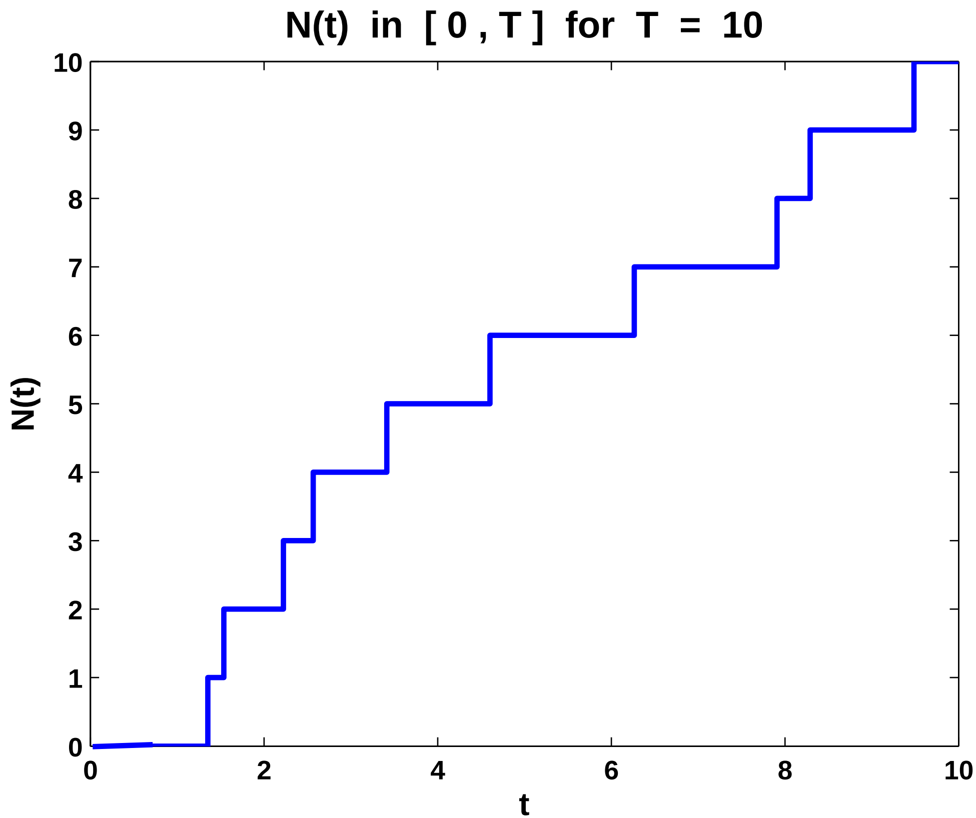
- ***Engineering*** perspective: applications to service systems

Illustration

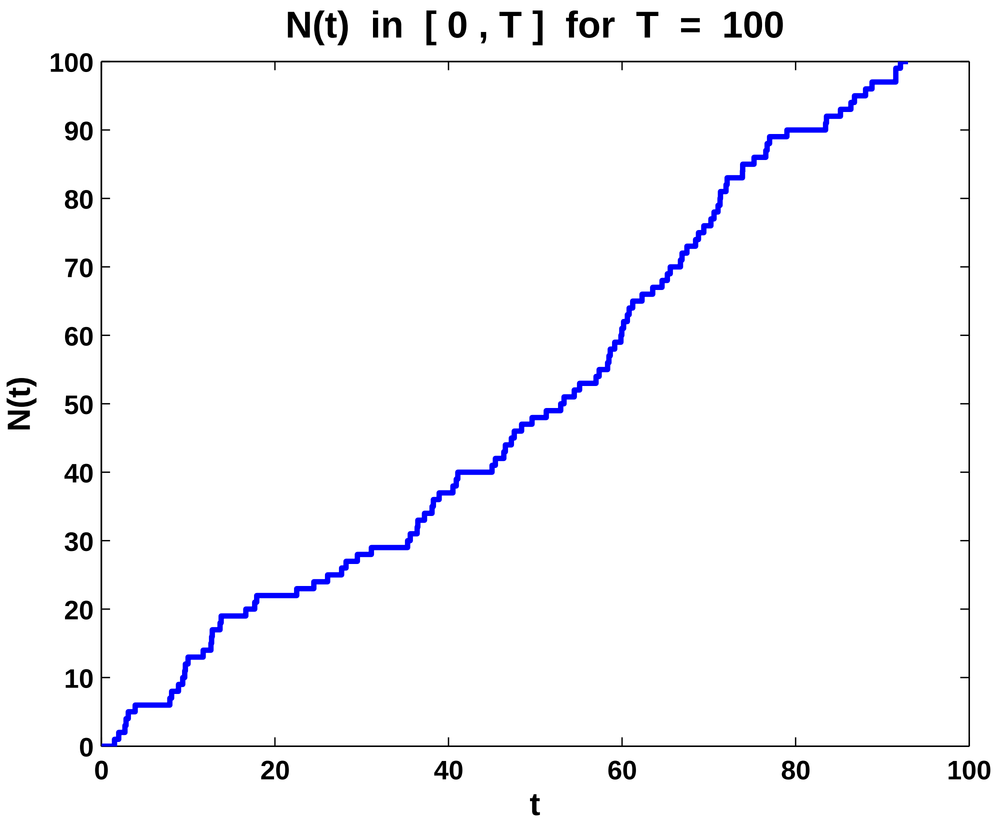
Unit Poisson Process

$N(t)$ is a Poisson (counting) process with rate 1

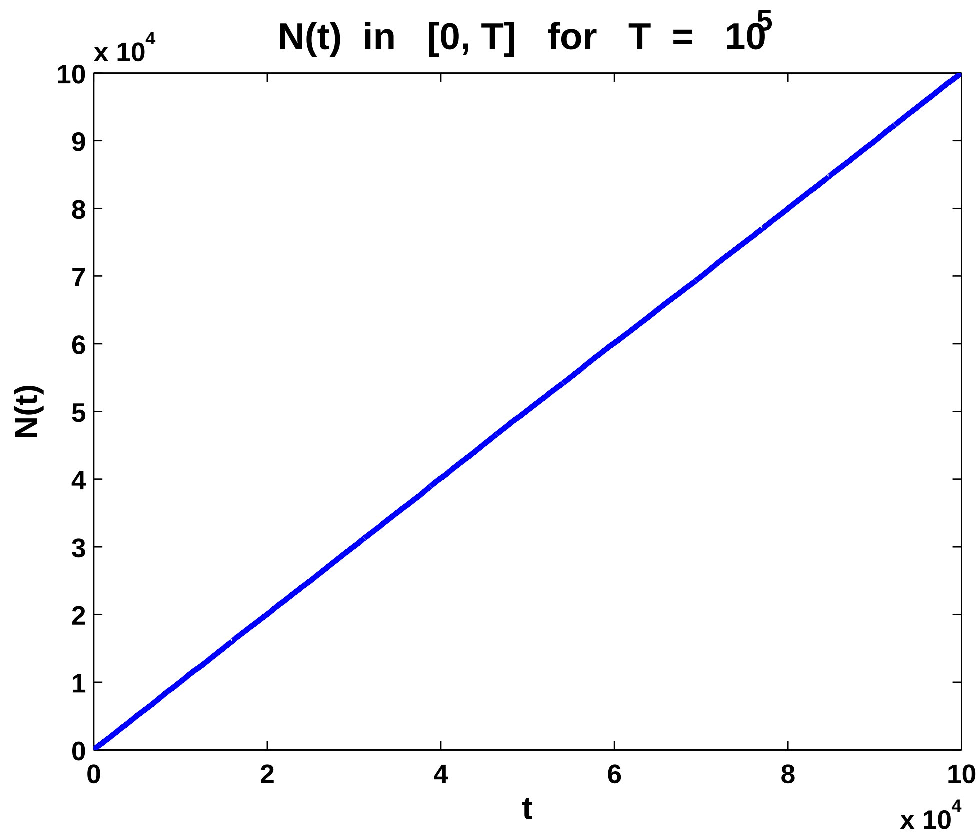
Plot $N(t)$ as a function of time t in $[0, T]$



N(t) in [0 , T] for T = 10
N(t)
t
0
1
2
3
4
5
6
7
8
9
10
0
2
4
6
8
10



N(t) in [0 , T] for T = 100
N(t)
t
0
10
20
30
40
50
60
70
80
90
100
0
20
40
60
80
100


N(t) in [0, T] for T = 10^5
x 10^4
N(t)
t
x 10^4
0
1
2
3
4
5
6
7
8
9
10
0
2
4
6
8
10

What's Going on?

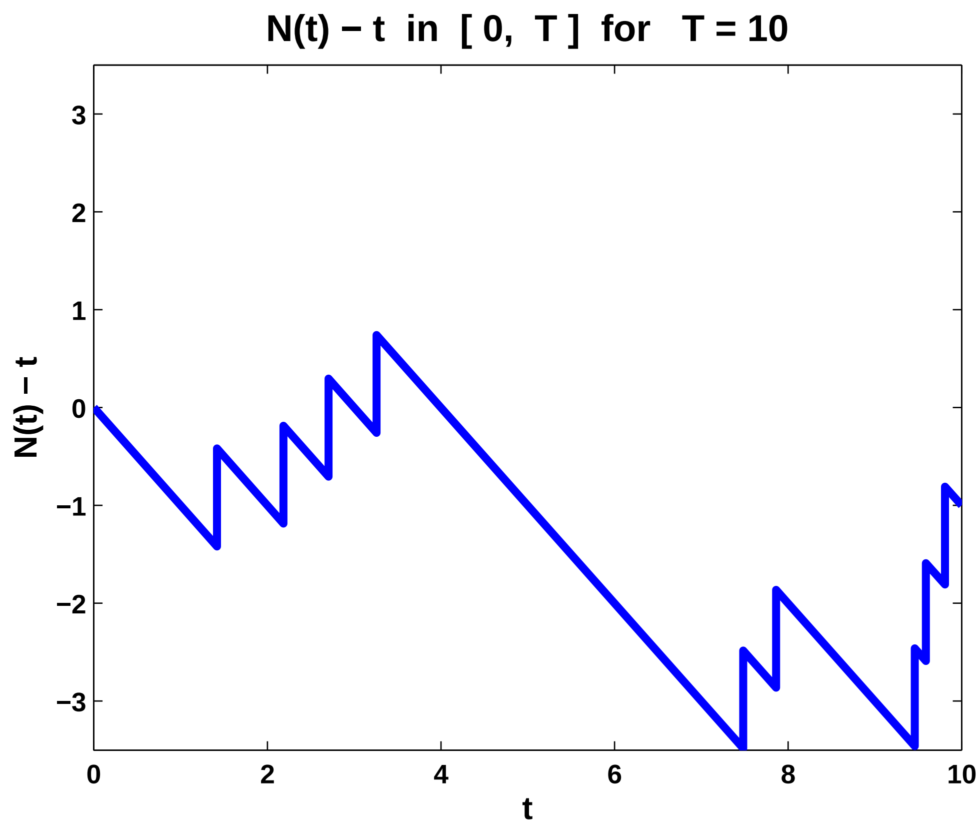
The plotter scales both ***time*** and ***space***.

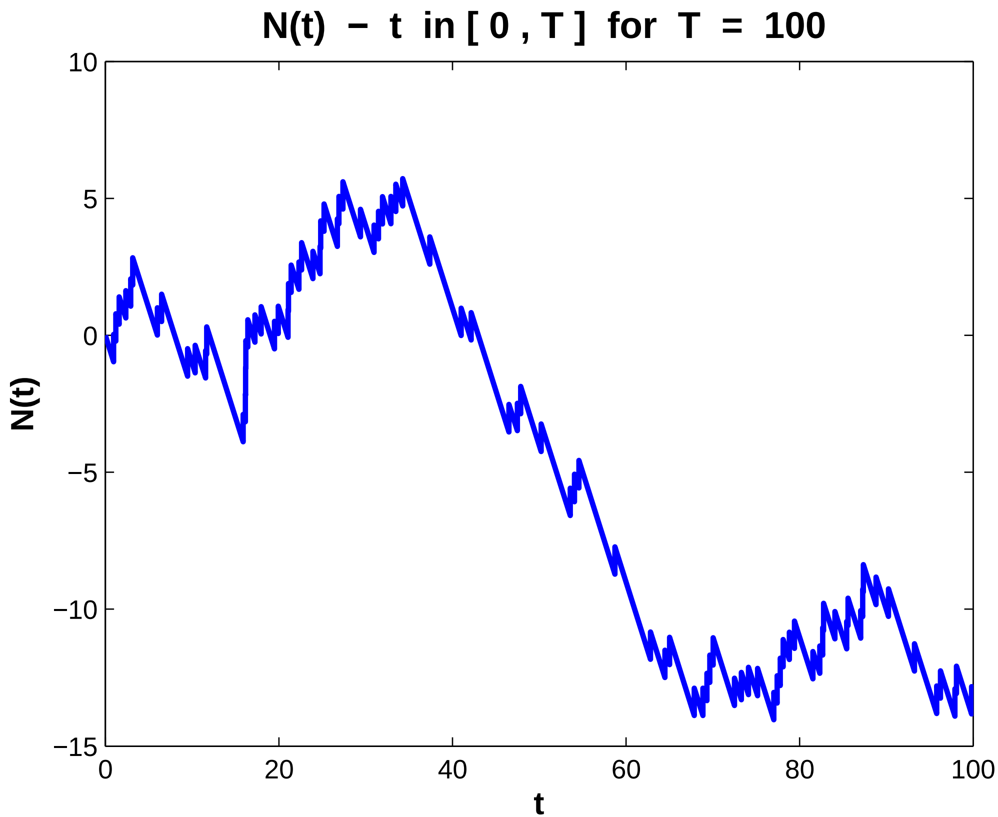
What we see is a sample path of:

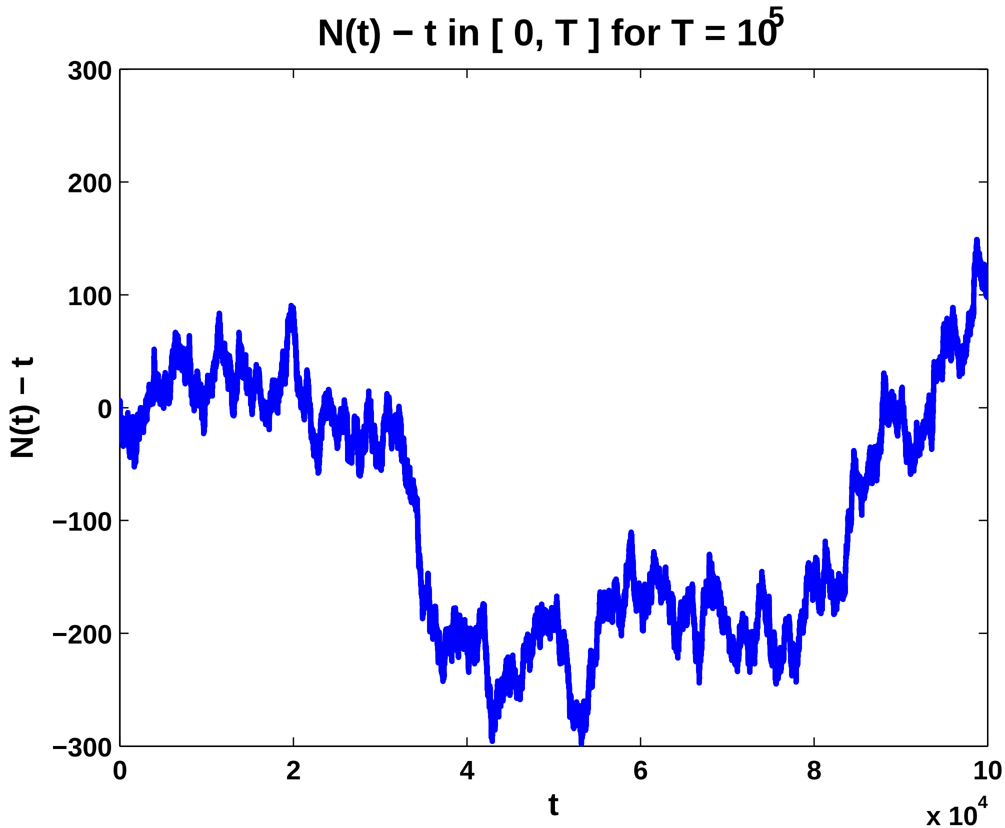
$$\bar{N}(t) = \frac{1}{n}N(nt) \text{ where } n = 10^4.$$

More Plots

Plot $N(t) - t$ as a function of time t in $[0, T]$


N(t) − t in [0, T] for T = 10
N(t) − t
t
3
2
1
0
−1
−2
−3
0
2
4
6
8
10


N(t) − t in [0 , T] for T = 100
N(t)
t
10
5
0
−5
−10
−15
0
20
40
60
80
100


N(t) − t in [0, T] for T = 10^5
300
200
100
0
−100
−200
−300
N(t) − t
0
2
4
6
8
10
t
x 10^4

What's Going on?

The plotter scales both ***time*** and ***space***.

What we see is a sample path of:

$$\hat{N}(t) = \frac{1}{\sqrt{n}}(N(nt) - nt) \text{ where } n = 10^4.$$

A Little Measure Theory

Measurable Space

$B(S)$ is a σ-algebra of subsets of S if:

1. $S \in B(S)$
2. If $F \in B(S)$ then $F^c \in B(S)$
3. If $F_1, F_2, F_3, \ldots \in B(S)$, then $\cup_i F_i \in B(S)$

$(S, B(S))$ ***is a measurable space***

Measure

- $\mu : (S, B(S)) \to [0, \infty]$ is a ***measure*** if $\mu(\phi) = 0$ and :

$$\mu(\cup_{i=1}^{\infty} F_i) = \Sigma_{i=1}^{\infty} \mu(F_i) ,$$

  where $(F_n : n \geq 1)$ is a sequence of pairwise disjoint sets in $B(S)$.

- μ is a ***probability measure*** if $\mu(S) = 1$.

Measurable Function

$h : (S_1, B(S_1)) \to (S_2, B(S_2))$ is ***measurable*** if:

$$h^{-1}(A) = \{s \in S_1 : h(s) \in A\} \in B(S_1),$$

for $A \in B(S_2)$.

Random Element

$$(\Omega, F, P) \xrightarrow{X} (S, B(S))$$

X is ***measurable***

- If $S \equiv \mathbf{R}$, then X is a ***random variable***
- If S is a function space (e.g., $\mathbf{D}$), then X is a ***stochastic process***

Stochastic Process Limits

Almost Sure Convergence

X_1, X_2, X_3, ... is a sequence of stochastic processes (e.g., in C).

$$X_n \to X \ \textbf{w.p.1} \Leftrightarrow P(\|X_n - X\|_T \to 0) = 1, \forall\, T \geq 0,$$

where,

$$||X_n - X||_T = \sup_{0 \leq t \leq T} |X_n(t) - X(t)| \ .$$

We say that X_n converges to X **a.s. u.o.c**.

Weak Convergence of Probability Measures

(S, m) metric space

$\{P_n : n \geq 1\}$ on (S, m) ***converges weakly*** to P if:

$$\lim_{n \to \infty} \int_S f dP_n = \int_S f dP, \ \forall f \in C(S).$$

And we write,

$$P_n \Rightarrow P \ .$$

Prohorov Topology

$P(S)$ is the space of probability measures on S.

We can define a metric π on $P(S)$, such that:

$$\pi(P_n, P) \to 0 \text{ in } P(S) \Leftrightarrow P_n \Rightarrow P, .$$

π is the ***Prohorov metric*** on $P(S)$.

Probability Law

Let $X : (\Omega, F, P) \longrightarrow (S, B(S))$ be a random element.

The ***probability law*** of X is defined as:

$$PX^{-1}(A) \equiv P(X^{-1}(A)) \equiv P(\{w \in \Omega : X(w) \in A\}),$$

for $A \in B(S)$.

Weak Convergence of Random Elements

Let $X_n, X : (\Omega, F, P) \to (S, B(S))$ be random elements.

We say that $X_n \Rightarrow X$ if

$$PX_n^{-1} \Rightarrow PX^{-1}$$

$$\Updownarrow$$

$$E[f(X_n)] \to E[f(X)], \forall f \in C(S).$$

A Useful Theorem

Skorohod Representation Theorem

If $X_n \Rightarrow X$ in (S, m), then there exists $\tilde{X}_n$ and $\tilde{X}$, ***defined on a common probability space*** such that:

$$\tilde{X}_n \sim X_n \text{ and } \tilde{X} \sim X \ ,$$

and

$$\tilde{X}_n \to \tilde{X} \text{ w.p.1.}$$

Proving Limits?

Continuous Mapping Approach

Simple Continuous-Mapping theorem (CMT)

If $X_n \Rightarrow X$ in (S, m) and $g : (S, m) \to (S', m')$ is ***continuous***, then:

$$g(X_n) \Rightarrow g(X) \text{ in } (S', m') \ .$$

Generalized Continuous Mapping Theorem

Let $g_n : (S, m) \longrightarrow (S', m')$ be ***measurable*** functions.

Suppose $g_n(x_n) \longrightarrow g(x)$ whenever $x_n \longrightarrow x$ for $x \in E \subseteq S$ such that $P(X \in E) = 1$

X_n, X random elements in (S, m).

If $X_n \Rightarrow X$ then $g_n(X_n) \Rightarrow g(X)$

Proof of Generalized Continuous Mapping Theorem

Theorem. Let $g_n : (S, m) \to (S', m')$ be ***measurable*** functions.

Suppose $g_n(x_n) \to g(x)$ whenever $x_n \to x$ for $x \in E \subseteq S$ such that $P(X \in E) = 1$

If $X_n \Rightarrow X$ then $g_n(X_n) \Rightarrow g(X)$.

Proof.
(***Skorohod***) $\Rightarrow \exists \tilde{X}_n \sim X_n, \tilde{X} \sim X$ *defined on the same space* such that:

$$\tilde{X}_n \to \tilde{X} \text{ w.p.1 },$$

where $P(\tilde{X} \in E) = 1$.

Deterministic Framework: $g_n(\tilde{X}_n) \to g(\tilde{X})$ w.p.1

$w.p.1$ convergence implies *weak* convergence: $g_n(\tilde{X}_n) \Rightarrow g(\tilde{X})$

By equality of distributions,

$$\boxed{g_n(X_n) \Rightarrow g(X).}$$

Another Useful Theorem

Donsker's Theorem

- $\{X_n : n \geq 1\}$ IID random variables
- $m = E[X_1]$ and $Var X_1 = \sigma^2 < \infty$

Define,

$$S_n(t) = \frac{1}{\sqrt{n}}(\sum_{i=1}^{\lfloor nt \rfloor} X_i - mnt), \quad t \geq 0 \ .$$

Then,

$$S_n \Rightarrow \sigma B \ ,$$

where B denotes standard Brownian motion.

Example: Renewal Process Asymptotics

A Fundamental Process

Define,

$$X(t) = \sum_{i=1}^{\lfloor t \rfloor} \xi_i, \; t \geq 0 \; ,$$

where $\{\xi_i, i \geq 1\}$ sequence of IID nonnegative random variables.

And the inverse process

$$Y(t) = \sup\{s \geq 0 : X(s) \leq t\} \; .$$

Then, $Y(t)$ is a renewal counting process!

Functional SLLN for Renewal Processes

Standard SLLN for X

$$\Downarrow$$

$$\bar{X}^n(t) \equiv \frac{1}{n} X(nt) \to \bar{X}(t) \equiv mt \text{ w.p.1 u.o.c } ,$$

$\Downarrow$ **(CMT)**

$$\bar{Y}^n(t) \equiv \frac{1}{n} Y(nt) \to \bar{Y}(t) \equiv t/m \text{ w.p.1 u.o.c } ,$$

where $m = E[\xi_i] > 0$ (finite).

Functional CLT for Renewal Processes

If,

$$\hat{X}^n(t) = \sqrt{n}(\bar{X}^n(t) - \bar{X}(t)) \ , \ t \geq 0,$$

$$\hat{Y}^n(t) = \sqrt{n}(\bar{Y}^n(t) - \bar{Y}(t)) \ , \ t \geq 0,$$

where $\bar{X}(t) = mt$ and $\bar{Y}(t) = t/m$.

Then,

$$\hat{X}^n(t) \Rightarrow \hat{X}(t) \equiv \sigma B(t)$$

$$\hat{Y}^n(t) \Rightarrow -\frac{1}{m}\hat{X}(t/m).$$

Random-Time Change Theorem

Random-Time Change Theorem

Let $\{U_n : n \geq 1\}$ and $\{V_n : n \geq 1\}$ be two sequences of stochastic processes in D.

If V_n is nondecreasing, $V_n \Rightarrow v$ (deterministic), and $U_n \Rightarrow U$, then,

$$U_n(V_n) \Rightarrow U(v) \ ,$$

where,

$$U_n(V_n) \equiv \{U_n(V_n(t)), t \geq 0\} \ ,$$

and,

$$U(v) \equiv \{U(v(t)), t \geq 0\} \ .$$

Proof of Functional CLT

Theorem. $\hat{X}^n(t) = \sqrt{n}(\bar{X}^n(t) - \bar{X}(t))$ and $\hat{Y}^n(t) = \sqrt{n}(\bar{Y}^n(t) - \bar{Y}(t))$
Then,
$\hat{X}^n(t) \Rightarrow \hat{X}(t) \equiv \sigma B(t)$ and $\hat{Y}^n(t) \Rightarrow -\frac{1}{m}\hat{X}(t/m)$

Proof.

Donsker's: $\hat{X}^n \Rightarrow \hat{X}$

FSLLN: $\bar{X}^n(t) \to \bar{X}(t)$ a.s. u.o.c. and $\bar{Y}^n(t) \to \bar{Y}(t)$ a.s. u.o.c

Random-Time Change: $\hat{X}^n(\bar{Y}^n) = \sqrt{n}[\bar{X}^n(\bar{Y}^n) - \bar{X}(\bar{Y}^n)] \Rightarrow \hat{X}(\bar{Y})$

Proof of Functional CLT (Cont')

Theorem. $\hat{X}^n(t) = \sqrt{n}(\bar{X}^n(t) - \bar{X}(t))$ and $\hat{Y}^n(t) = \sqrt{n}(\bar{Y}^n(t) - \bar{Y}(t))$
Then,
$\hat{X}^n(t) \Rightarrow \hat{X}(t) \equiv \sigma B(t)$ and $\hat{Y}^n(t) \Rightarrow -\frac{1}{m}\hat{X}(t/m)$

Proof (Cont').

$$
\begin{aligned}
\hat{Y}^n &= \sqrt{n}(\bar{Y}^n - \bar{Y}) \\
&= \frac{\sqrt{n}}{m}[\bar{X}(\bar{Y}^n) - \bar{X}(\bar{Y})] \\
&= \frac{\sqrt{n}}{m}[\bar{X}(\bar{Y}^n) - \bar{X}^n(\bar{Y}^n) + \bar{X}^n(\bar{Y}^n) - \bar{X}(\bar{Y})] \\
&= \frac{\sqrt{n}}{m}[\bar{X}(\bar{Y}^n) - \bar{X}^n(\bar{Y}^n)] + \frac{\sqrt{n}}{m}[\bar{X}^n(\bar{Y}^n) - \bar{X}(\bar{Y})] \\
&\Rightarrow -\frac{1}{m}\hat{X}(\bar{Y}) ,
\end{aligned}
$$

since we can show that $\sqrt{n}[\bar{X}^n(\bar{Y}^n) - \bar{X}(\bar{Y})] \longrightarrow 0$ a.s. u.o.c.

Thanks!