Assignment 3: Discrete-Time Markov Chain

– due September 11

Do Exercises 4.3, 4.7, 4.8 in Green Ross.

Other problems:

0. **(Conditional Bayes' rule and law of total probability)**

   Consider events A, B and C and a partition of the sample space $B_1, \dots, B_n$. Show:

   $$(a) \qquad \mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\,\mathbb{P}(B|C) \qquad \text{(conditional Bayes' rule)} \tag{1}$$

   $$(b) \qquad \mathbb{P}(A|C) = \sum_{i=1}^{n} \mathbb{P}(A|B_k \cap C)\,\mathbb{P}(B_k|C). \qquad \text{(conditional LOTP)} \tag{2}$$

   Remark: Removing C, (1) and (2) will degenerate to the regular Bayes' rule and LOTP.

1. **(DTMC: basic concept)** Let $X_n = Y_{n-1} + Y_n$, $n \geq 1$, $X_0 = 0$, where $\{Y_n : n \geq 1\}$ is an I.I.D. sequence of Bernoulli random variables: $\mathbb{P}(Y = 0) = \mathbb{P}(Y = 1) = 0.5$. Is $\{X_n : n \geq 1\}$ a Markov chain?

2. **(DTMC: formulation)** A store that stocks a certain commodity uses the following (s, S) ordering policy: if its inventory at the beginning of a day is x, then it orders

   $$\begin{aligned} 0 \quad & \text{if } x \geq s, \\ S - x \quad & \text{if } x < s. \end{aligned}$$

   In other words, if the current inventory level is above s, do not order; if it is below s, order up to S. Assume no lead time, i.e., the order is immediately filled. The daily demands are I.I.D. random variables, with a PMF $P(j) = \alpha_j$, $j \geq 0$. We assume that all demands that cannot be immediately met are lost, i.e., there is no backlog. Let X_n be the inventory level at the end of day n. Is the process $\{X_n, n \geq 1\}$ a Markov Chain? If yes, find its state space and its transition probabilities.

3. **(DTMC: advanced Markov property)** Consider a DTMC $\{X_n, n \geq 0\}$, prove that

   $$\mathbb{P}(X_n = j|X_{n_1} = i_1, \dots, X_{n_k} = i_k) = \mathbb{P}(X_n = j|X_{n_k} = i_k), \text{for } 0 \leq n_1 < n_2 < \cdots < n_k < n. \tag{3}$$

   Remark: When we proved the C-K equation in class, we have already used (3), which means that in order to predict the future (i.e., what happens on day n, that is X_n), given the latest past (i.e., information on day n_k, that is X_{n_k}), we don't need any older information before (i.e., from day n_1 to day n_{k-1}).
   Hint: Start with the case $n = n_k + 1$, which is automatic by the Markov property, then generalize it to the case $n = n_k + j$ for all $j \geq 1$ using induction. In addition, use (1).

Reading: Read Section 4.1 in Chapter 4, do Exercises 4.1 and 4.4 in Ross, they are good examples and have answers in the back of the book.