APPENDIX
to
Stabilizing Performance in a Network of Many-Server Queues with Time-Varying Arrival Rates

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Abstract

This paper extends to feed-forward networks of many-server queues previous modified-offered-load staffing algorithms to stabilize abandonment probabilities and expected delays for a single many-server queue with time-varying arrival rates and customer abandonment, emphasizing the challenging case of longer service times. The network algorithm is also successful in stabilizing performance across a wide range of Quality of Service (QoS) targets, provided that the targeted QoS is either consistently high or consistently low at all queues. A limit theorem is established showing that the algorithm achieves its objectives asymptotically as the scale increases. However, it is shown that the performance of the network algorithm for fixed scale degrades significantly at a queue with a high QoS target if a previous queue feeding it has a low QoS target and a service-time distribution that is not nearly exponential, because the departure process from that previous queue is not nearly a nonhomogeneous Poisson process (NHPP), as required for the modified-offered-load refinement. Statistical tests based on the index of dispersion for counts are shown to be effective to determine if arrival processes are nearly NHPP's. Simulation experiments are conducted to verify the effectiveness of the staffing algorithm and the statistical tests of the departure processes.

1 Overview

This is an appendix to the main paper [3], providing additional supplementary material. In §2 we present additional results from simulation experiments, supplementing §5 of the main paper. In §3 we provide additional analysis of departure processes, supplementing §4 of the main paper.

2 Additional Experiments

In this section we supplement the main paper by presenting additional results from simulation experiments. We start in §2.1 by considering an example with much smaller external arrival rate, with $\bar{\lambda}$ reduced from 100 to 20. Then in §2.2 we consider additional examples with lognormal service-time distributions. In §2.3 we consider an example of a more general feed-forward network example with the departure process split.

2.1 Lower Arrival Rates and Staffing

We now supplement §5.1 by by showing in Figures 1 and 2 the analog of Figures 2 and 4 for the same model except $\bar{\lambda}$ is reduced from 100 to 20. As the scale decreases, the discretization becomes a more and more serious issue. Thus there is a limit to the stabilization that can be achieved with very small scale. Here we increase the number of replications to 5000.

Figure 1: Performance functions in the $M_t/H_2/s_t + M$ network with the sinusoidal arrival rate in (5) for $r = 0.4$ and $\bar{\lambda} = 20$: the cases of low QoS targets ($\alpha = 0.05$, 0.1, 0.15 and 0.2) and DIS staffing at both queues.

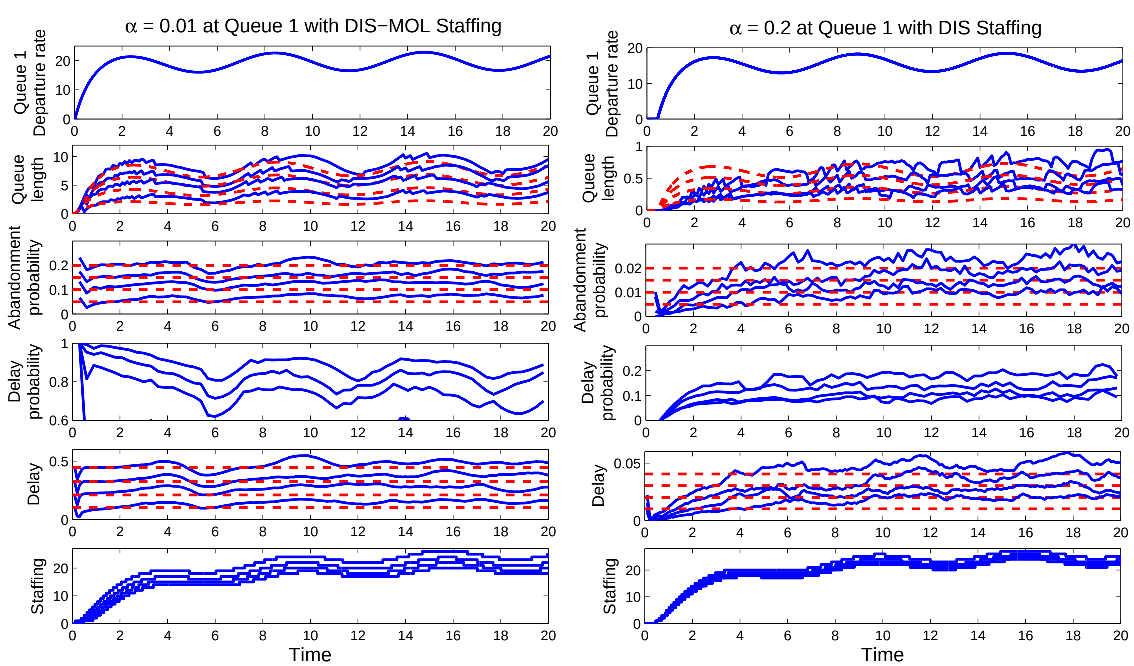
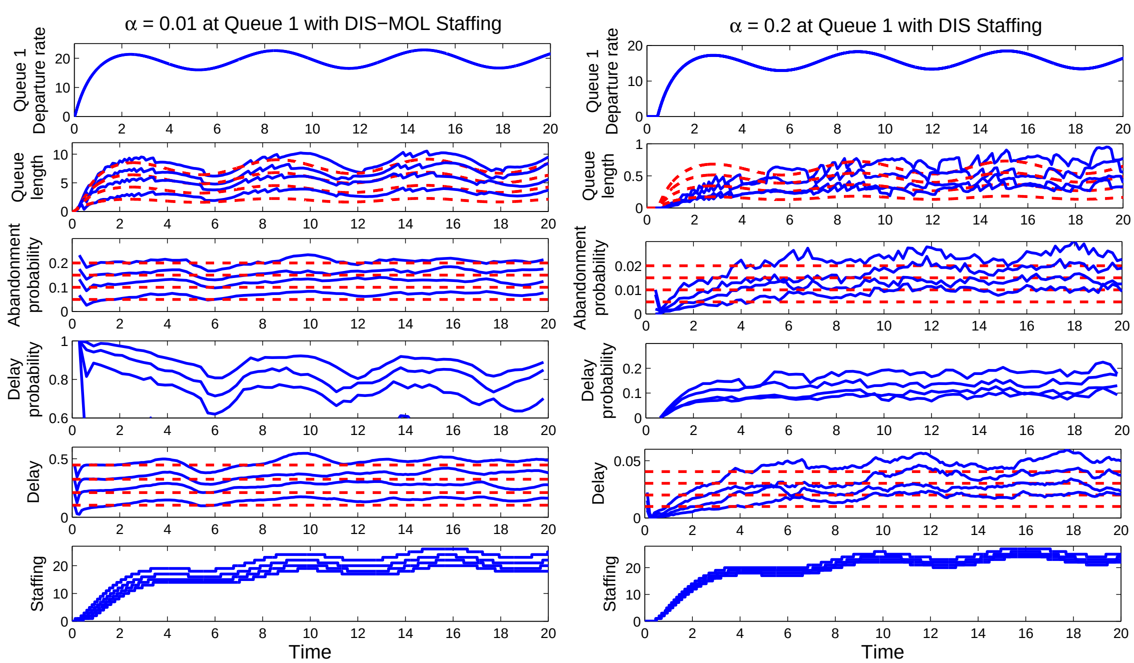


Figure 2: Performance functions in the $M_t/H_2/s_t + M$ network with the sinusoidal arrival rate in (5) for $r = 0.4$ and $\bar{\lambda} = 20$: the cases of low QoS targets ($\alpha = 0.05$, 0.10, 0.15 and 0.2) and DIS staffing at the second queue with fixed target $\alpha = 0.010$ at the first queue on the left, and high QoS targets ($\alpha = 0.005$, 0.010, 0.015 and 0.020) and DIS-MOL staffing at the second queue with fixed target $\alpha = 0.20$ at the first queue on the right.



2.2 Lognormal Service-Time Distribution

We now supplement §5.4 by by showing in Figures 3 and 4 the analog of Figures 2 and 4 for the $M_t/LN/s_t + M$ with $c^2 = 4$ and other parameters in §5.4.

Figure 3: Performance functions in the $M_t/LN/s_t + M$ network with the sinusoidal arrival rate in (5) for $r = 0.4$ and $\bar{\lambda} = 100$ when the service-time distribution is lognormal (LN) with scv $c^2 = 4$ at both queues: the case of low QoS targets ($\alpha = 0.05$, 0.1, 0.15 and 0.20) and DIS staffing at both queues.

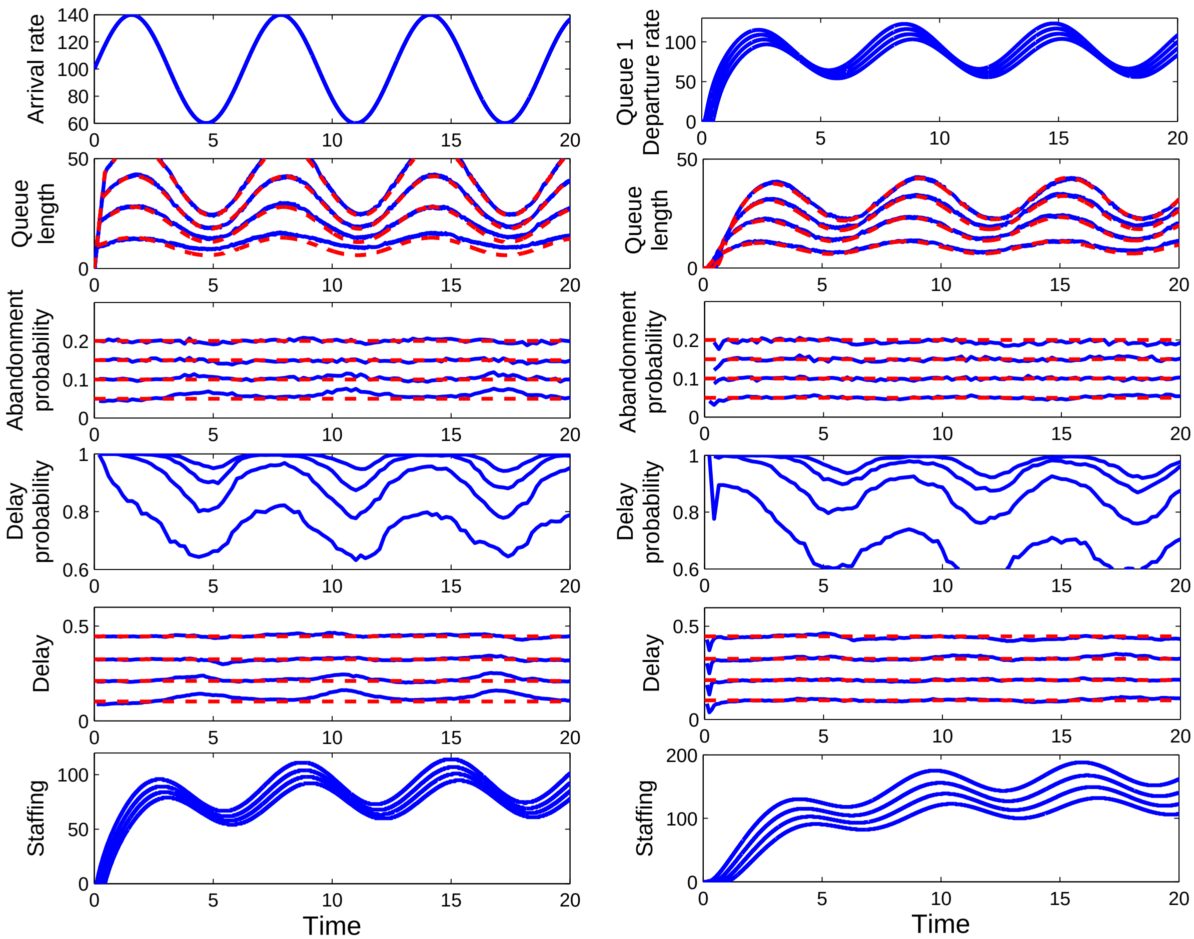
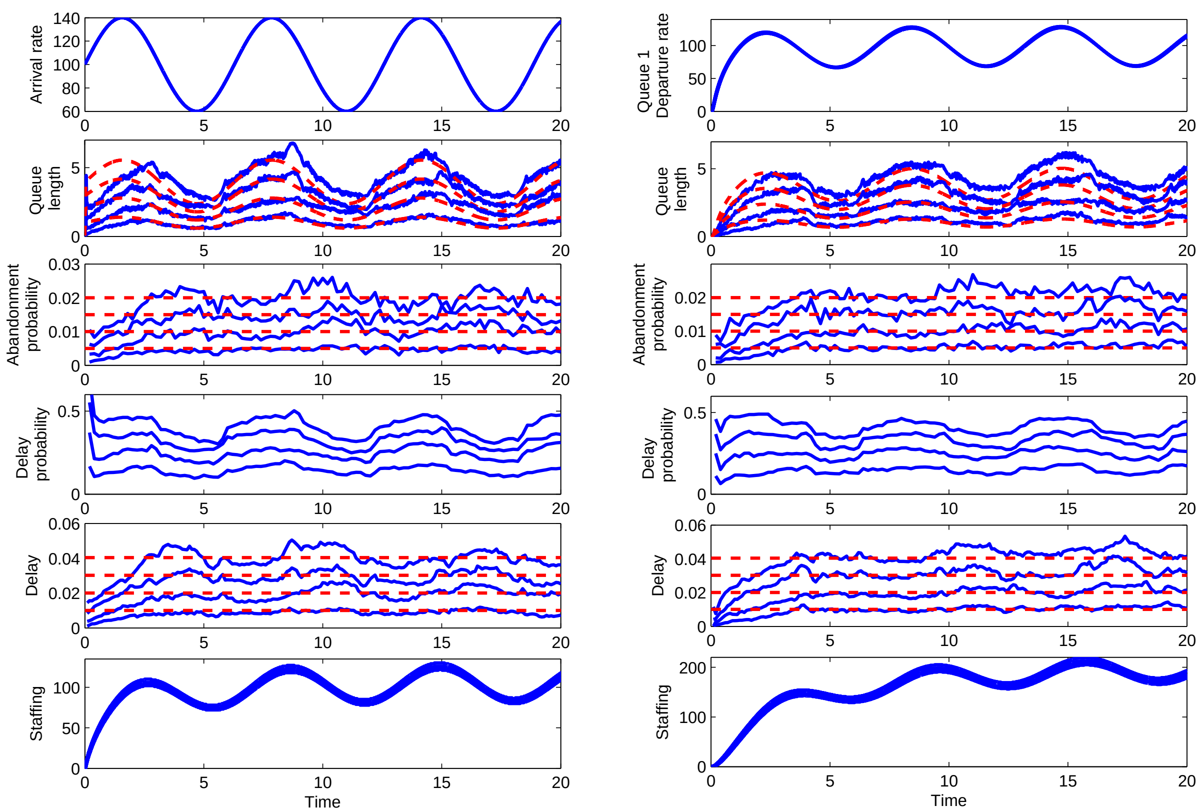


Figure 4: Performance functions in the $M_t/LN/s_t + M$ network with the sinusoidal arrival rate in (5) for $r = 0.4$ and $\bar{\lambda} = 100$ when the service-time distribution is lognormal (LN) with scv $c^2 = 4$ at both queues: the case of low QoS targets ($\alpha = 0.005$, 0.01, 0.015 and 0.02) and DIS-MOL staffing at both queues.



2.3 Three-Queue Feed-forward Model with Splitting

We finally experiment on a more general feed-forward network with three queues and splitting. To incorporate the Markovian routing structure, depicted in Figure 5, we consider three $M_t/H_2/s_t+M$ queues, with Queue 1 feeding Queues 2 and 3 with probabilities $p_{1,2} = 1 - p_{1,3} = 0.6$. Each queue has an H_2 service-time distribution with mean 1 and $c^2 = 4$. The arrival rate function is again sinusoidal as in (5) with relative amplitude $r = 0.4$. Figures 6 and 7 show good performance of DIS (for low QoS) and DIS-MOL (for high QoS), that are consistent with Figures 2 and 3 before.

Figure 5: The feedforward queueing network with Markovian routing.

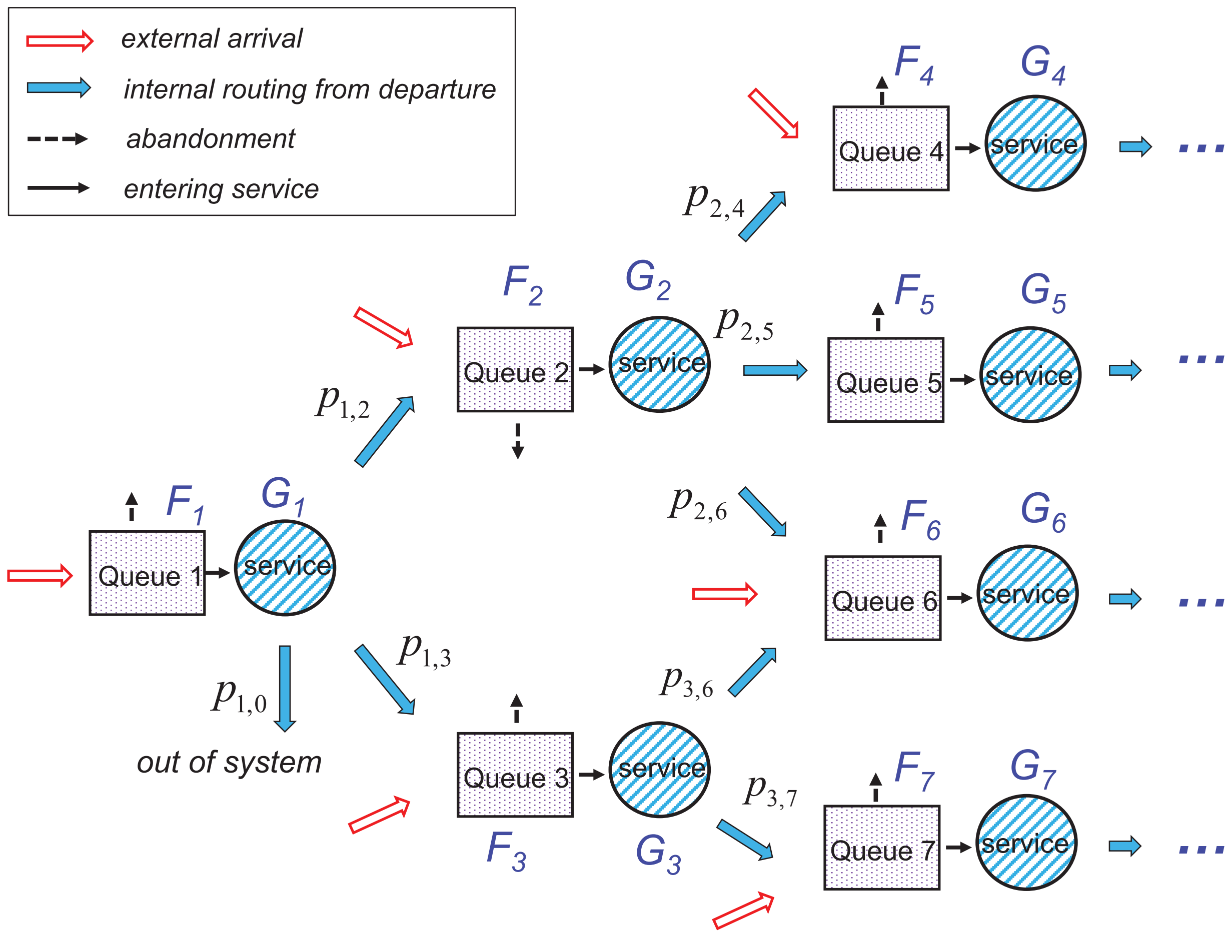


Figure 6: Performance functions in a three-queue distribution network of $M_t/H_2/s_t + M$ queues, with sinusoidal arrival rate in (5) for $r = 0.4$, mean service times 1 and routing probabilities $p_{1,2} = 1 - p_{1,3} = 0.6$: the cases of identical low QoS targets ($\alpha = 0.05$, 0.10, 0.15 and 0.20) and simple DIS staffing at all queues.

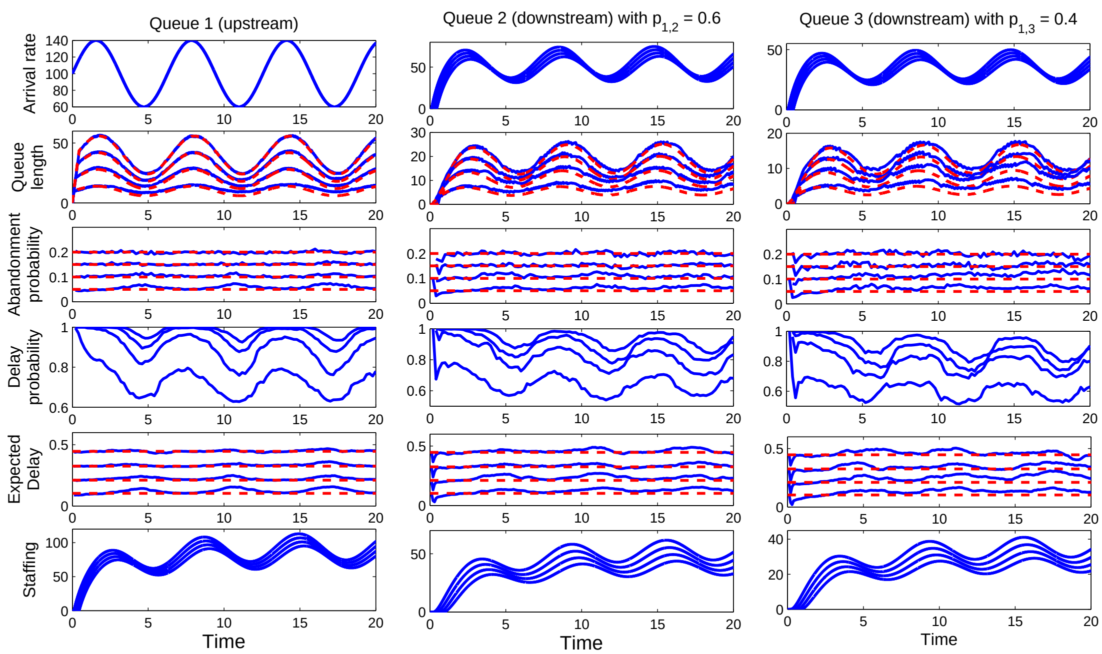
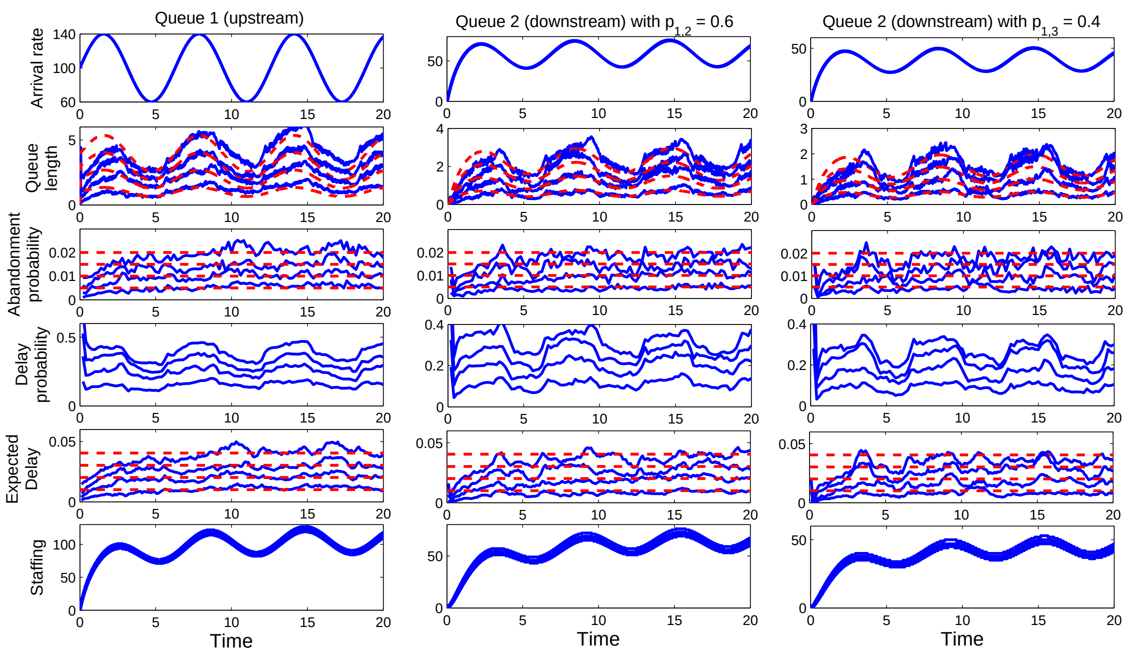


Figure 7: Performance functions in a three-queue distribution network of $M_t/H_2/s_t + M$ queues, with sinusoidal arrival rate in (5) for $r = 0.4$, mean service times 1 and routing probabilities $p_{1,2} = 1 - p_{1,3} = 0.6$: the cases of identical high QoS targets ($\alpha = 0.005$, 0.010, 0.015 and 0.020) and DIS-MOL staffing at all queues.



3 Direct Analysis of Departure Process Simulation Data

In this section we supplement §4 of the main paper by presenting additional analyses of departure process data from the simulation experiments. We start in §3.1 by presenting additional histograms of interdeparture times from the stationary $M/H_2/s+M$ model, showing that they tend to be exponentially distributed even if the departure process is not nearly a Poisson process (as revealed by the IDC and the performance in a following queue.) Next in §3.2 to test the approximation of the departure rate function, showing that it is quite good. Finally, in §3.3 we present additional results of Kolmogorov-Statistical tests of an NHPP applied to the departure process data, drawing upon [1, 2].

3.1 Interdeparture-Time Distribution for the $M/H_2/s+M$ Model

Complementing our analysis on the interdeparture times from the stationary $M/H_2/s+M$ model, we provide the histograms, fitted densities and hazard rate functions with a wider range of the abandonment probability targets: $\alpha = 0.3, 0.4$ and 0.5 in Figure 8 and $\alpha = 0.005, 0.01$ and 0.02 in Figure 9.

Figure 8: Histograms of the interdeparture times from the stationary $M/H_2/s+M$ model (on the left) and fitted densities and hazard rate functions (on the right): the cases of low QoS targets and DIS staffing: a range of abandonment probability targets $\alpha = 0.3, 0.4, 0.5$.

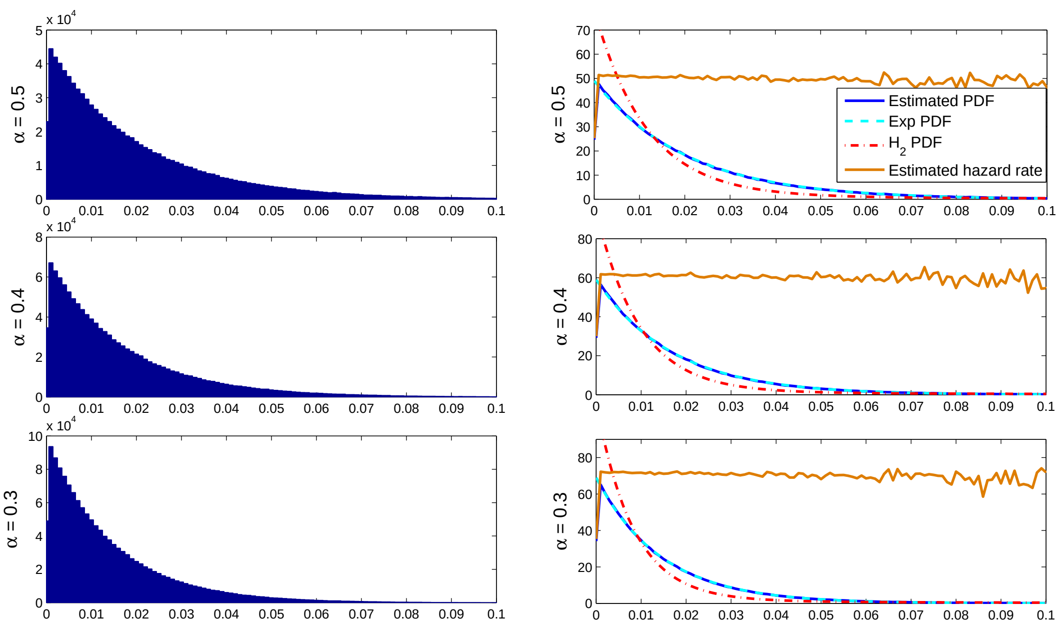
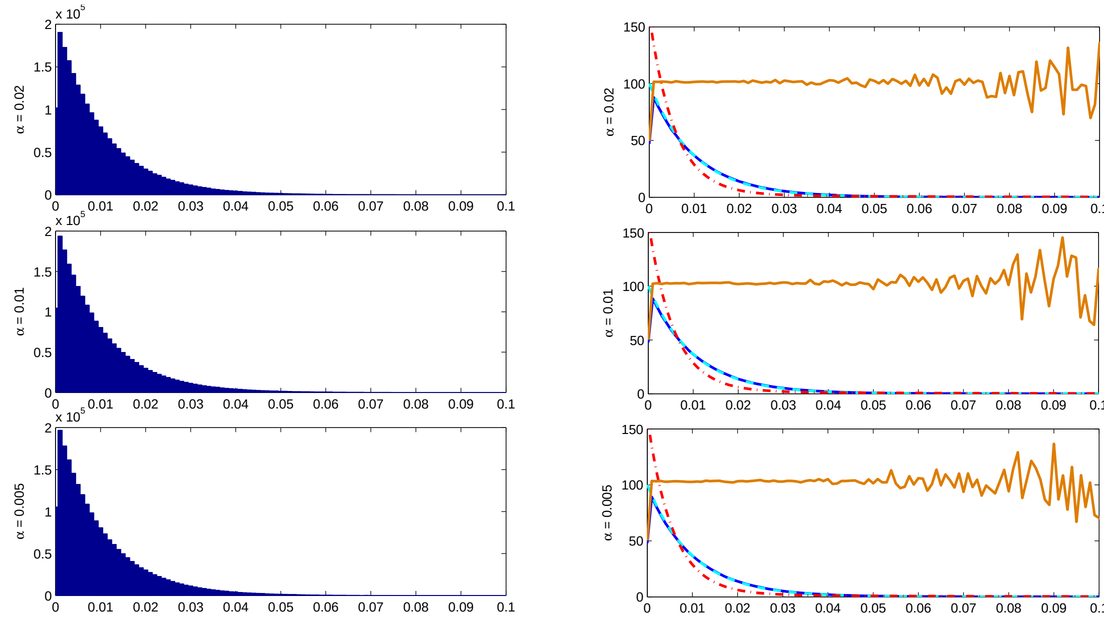


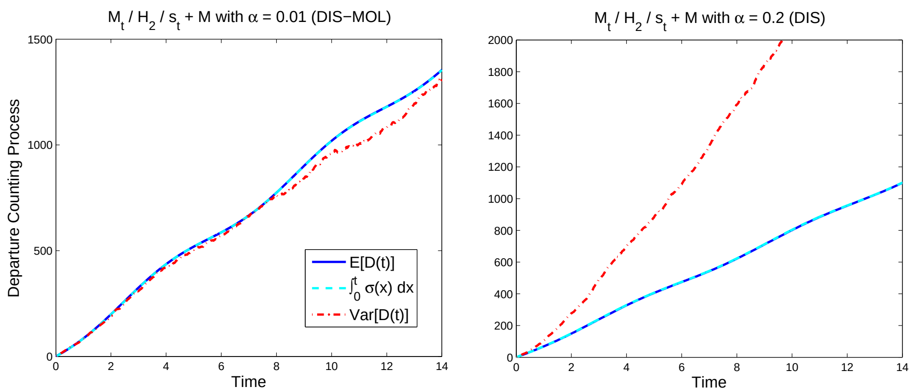
Figure 9: Histograms of the interdeparture times from the stationary $M/H_2/s+M$ model (on the left) and fitted densities and hazard rate functions (on the right): the cases of high QoS targets and DISMOL staffing: a range of abandonment probability targets $\alpha = 0.005, 0.01, 0.02$.



3.2 Approximating the Departure Rate

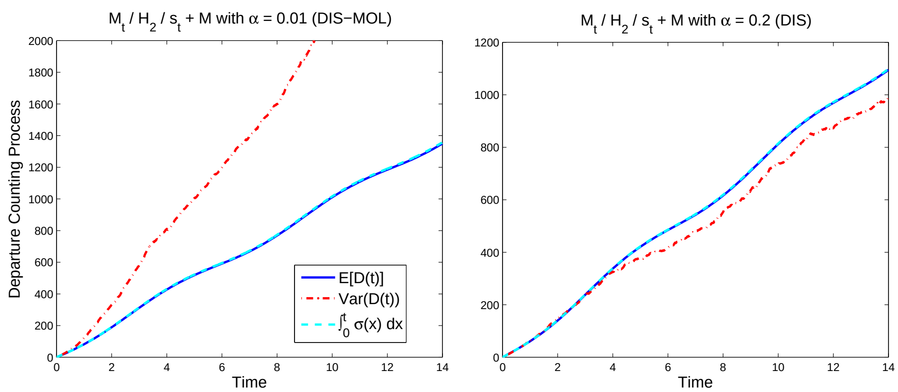
We next test the approximation quality for the departure rate function from the $G_t/GI/s_t + GI$ model under both DIS (with low QoS) and DISMOL (with high QoS) staffing functions. We plot the simulation estimations for the means (solid lines) and variances (dashed-and-dotted lines) of the cumulative departure counting process for two models (i) $M_t/H_2/s_t + M$, in Figure 10 and (ii) $H_2(t)/M/s_t + M$, in Figure 11. We also compare the estimated means to the approximating DIS cumulative departure rates $\Sigma(t) \equiv \int_0^t \sigma_1(x)\,dx$ (the dashed lines), where σ_1 is the approximating departure rate function from the first queue given in Corollary 1 of the main paper.

Figure 10: Mean and variance of $D(t)$, the cumulative number of departures in $[0, t]$, for the departure process from the $M_t/H_2/s_t + M$ model with sinusoidal arrival rate in (5) for $r = 0.4$: $\alpha = 0.01$ (left-hand plot) and $\alpha = 0.2$ (right-hand plot).



We make two observations. First, the estimated mean of the departure counting processes essentially coincides with the approximating DIS cumulative departure rate function Σ_1. Second, the two models $M_t/H_2/s_t+M$ and $H_2(t)/M/s_t+M$ exhibit quite different (opposite) performance: the departure process from $M_t/H_2/s_t + M$ ($H_2(t)/M/s_t + M$) is close to an NHPP for high (low) QoS target but not for low (high) QoS target. To understand the somewhat counterintuitive performance in Figure 11, we note that the departure process becomes an NHPP for low QoS target because almost all the servers are busy at all times. However, due to the non-M_t arrival assumption, the departure process, which is close to an $H_2(t)/M/\infty$ queue for high QoS targets, can be far from an NHPP.

Figure 11: Mean and variance of $D(t)$, the cumulative number of departures in $[0, t]$, for the departure process from the $H_2(t)/M/s_t + M$ model with sinusoidal arrival rate in (5) for $r = 0.4$: $\alpha = 0.01$ (left-hand plot) and $\alpha = 0.2$ (right-hand plot).



3.3 Kolmogorov-Statistical Tests of an NHPP

In this section we apply the Kolmogorov-Smirnov (KS) statistical tests of an NHPP from [1, 2] to the departure process data obtained from the simulation experiments. We first present direct estimates of the departure rate functions for the difference examples. Then we present results of KS tests.

Figure 12: Estimated departure rate function from the stationary $M/H_2/s+M$ queue with constant arrival rate $\lambda = 100$ with high QoS (low abandonment probability) targets, starting out empty: based on 1000 replications using subintervals of length 0.1. 95% confidence intervals bands are shown.

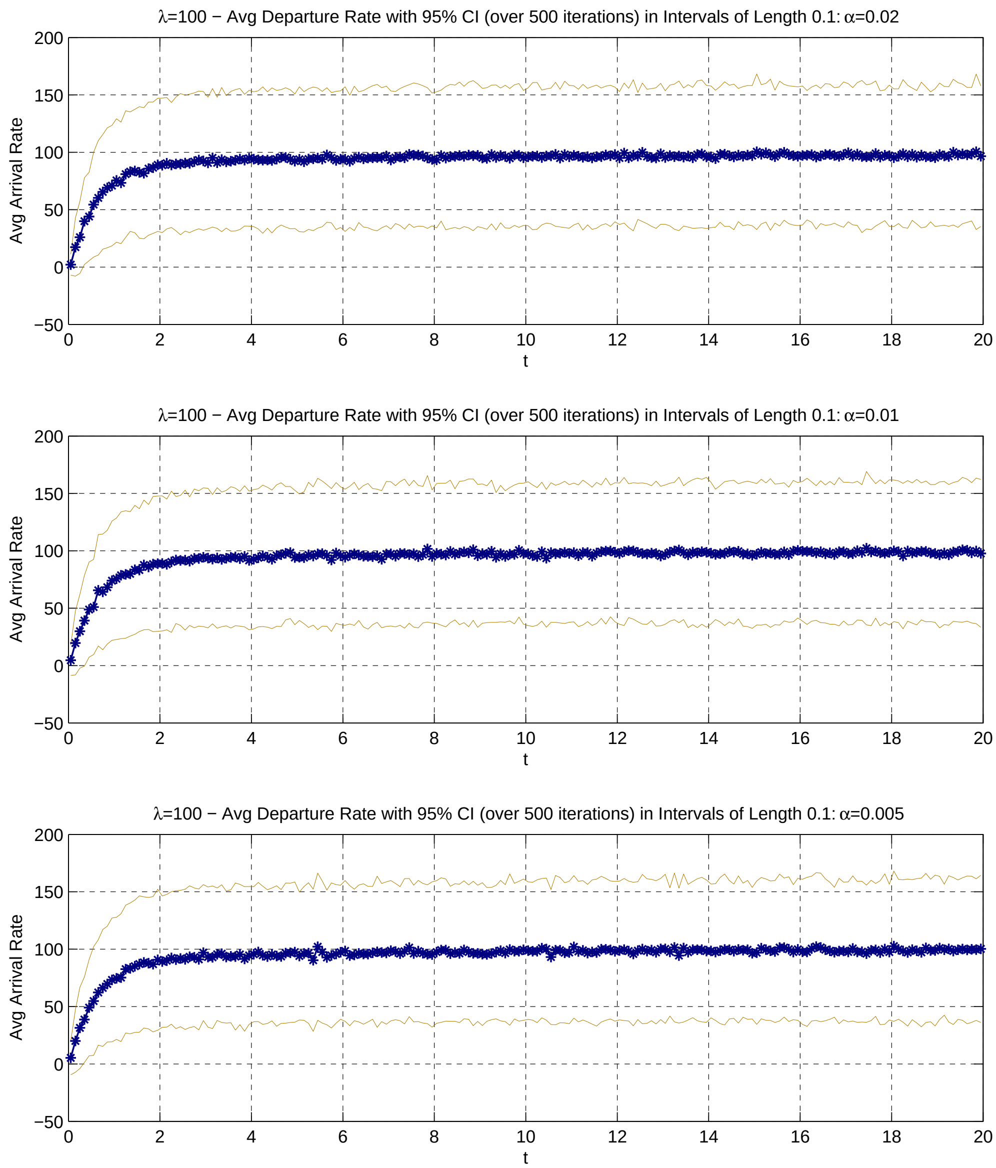


Figure 13: Estimated departure rate function from the stationary $M/H_2/s+M$ queue with constant arrival rate $\lambda = 100$ with low QoS (high abandonment probability) targets, starting out empty: based on 1000 replications using subintervals of length 0.1. 95% confidence intervals bands are shown.

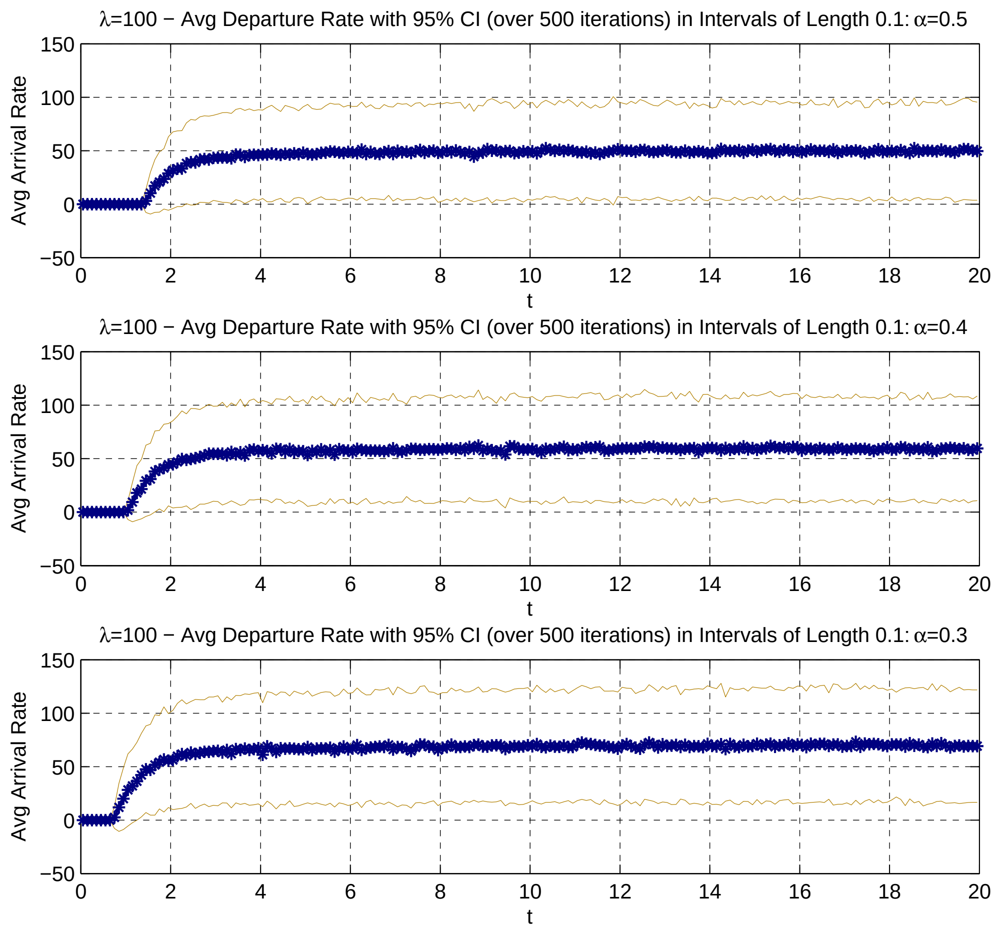


Figure 14: Estimated departure rate function from the $M_t/H_2/s+M$ queue with sinusoidal arrival rate $\lambda(t) = 100 + 20sin(t)$ with high QoS (low abandonment probability) targets, starting out empty: based on 1000 replications using subintervals of length 0.1. 95% confidence intervals bands are shown.

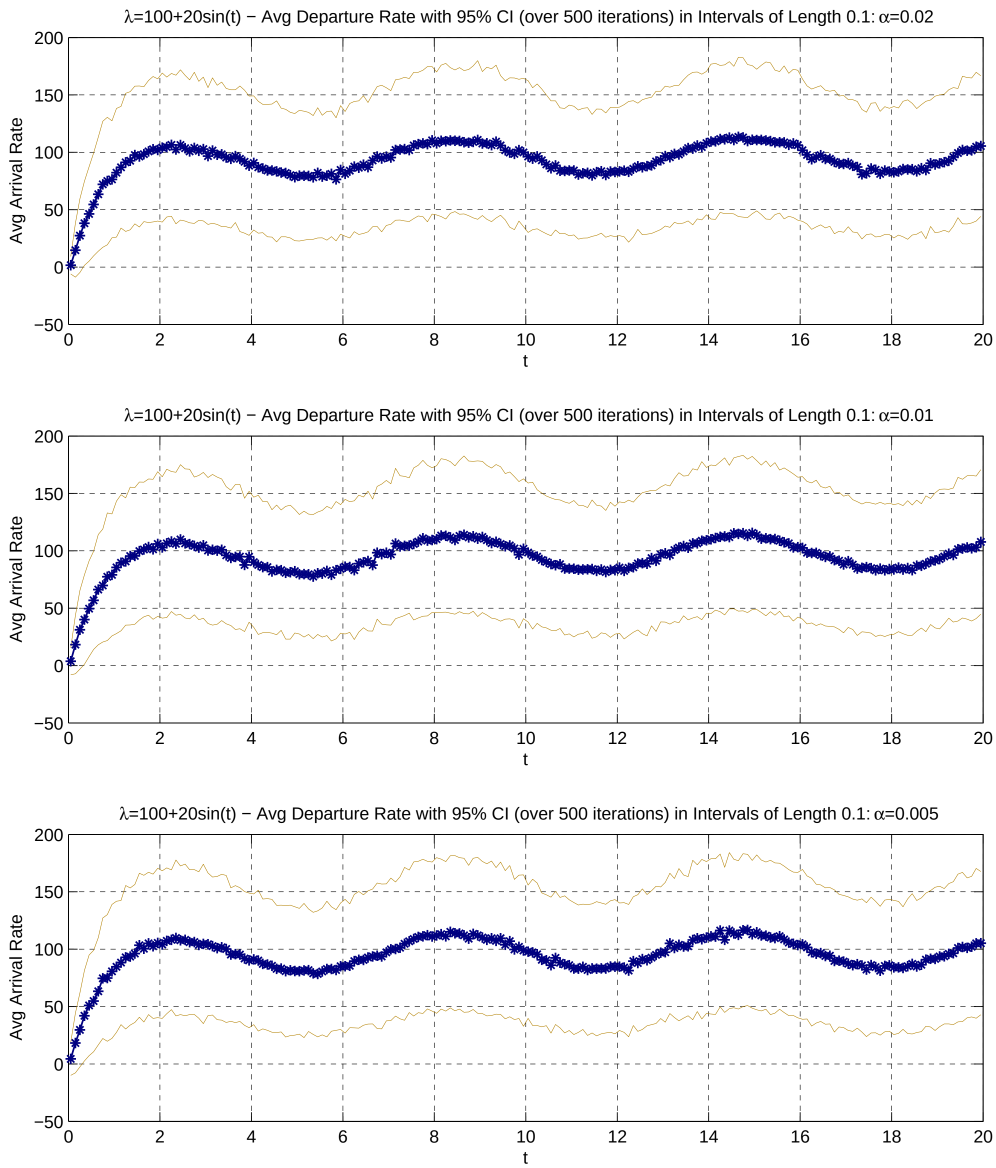


Figure 15: Estimated departure rate function from the $M_t/H_2/s+M$ queue with sinusoidal arrival rate $\lambda(t) = 100 + 20sin(t)$ with low QoS (hig abandonment probability) targets, starting out empty: based on 1000 replications using subintervals of length 0.1. 95% confidence intervals bands are shown.

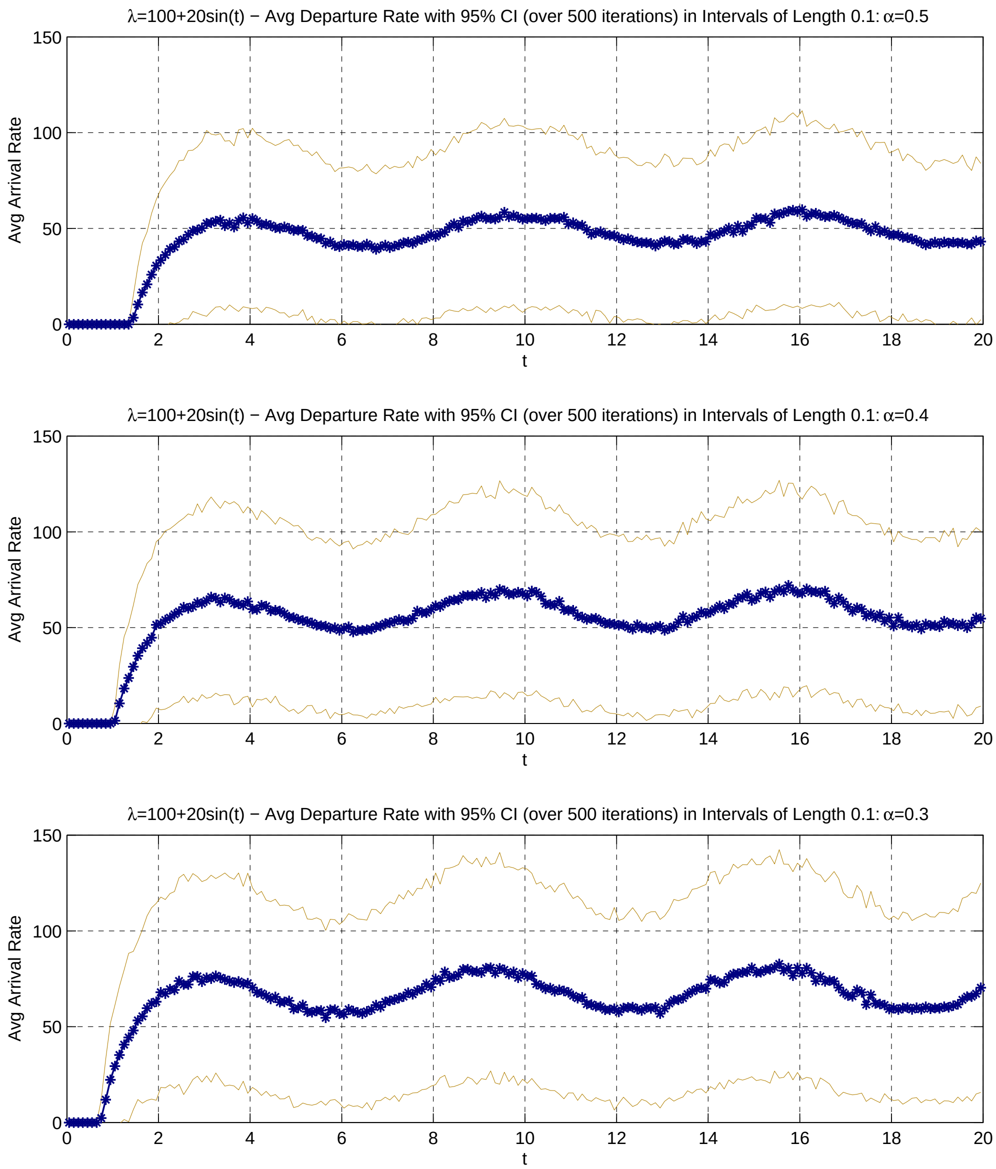


Figure 16: Estimated departure rate function from the $M_t/H_2/s+M$ queue with sinusoidal arrival rate $\lambda(t) = 100 + 60sin(t)$ with high QoS (low abandonment probability) targets, starting out empty: based on 1000 replications using subintervals of length 0.1. 95% confidence intervals bands are shown.

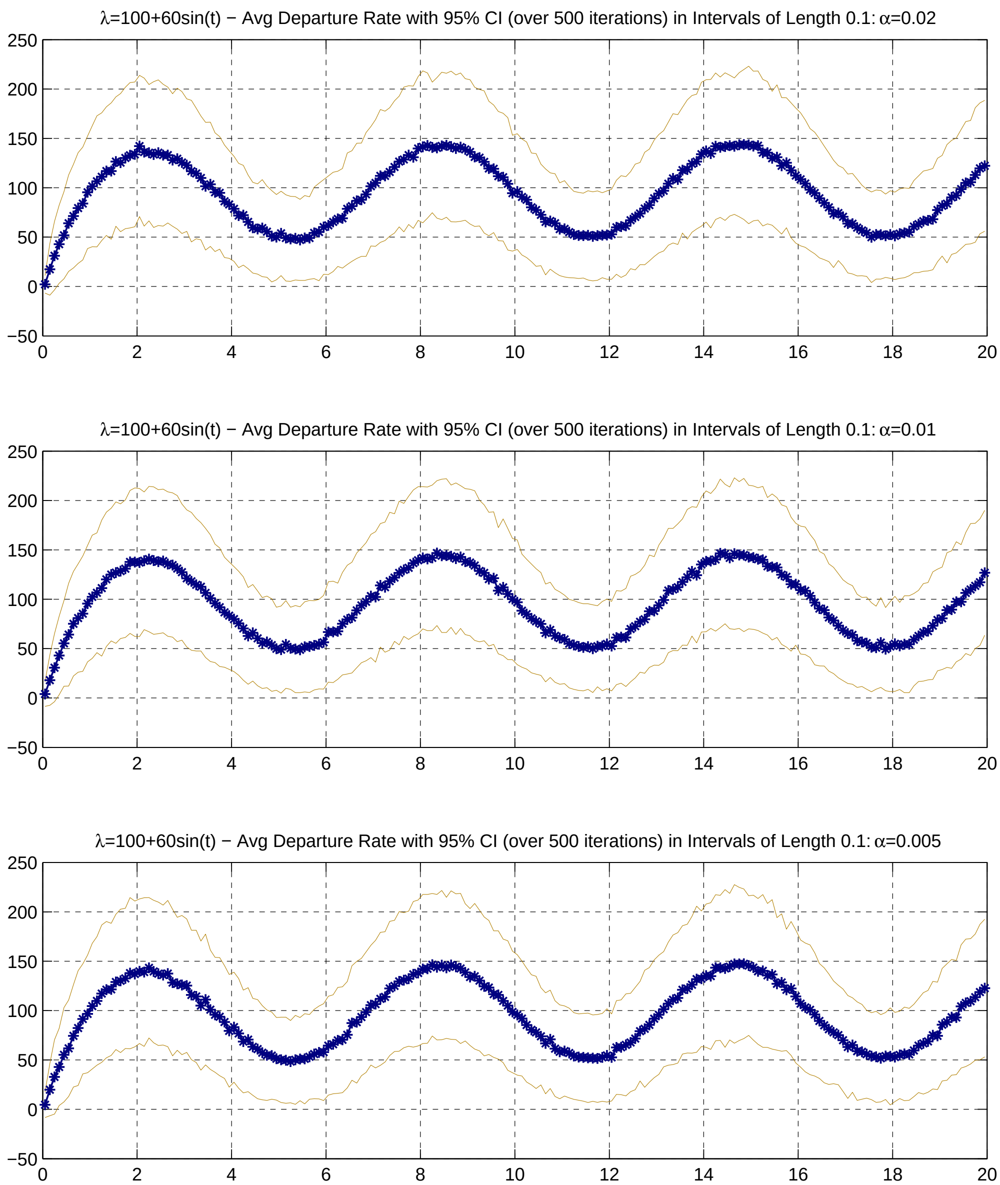
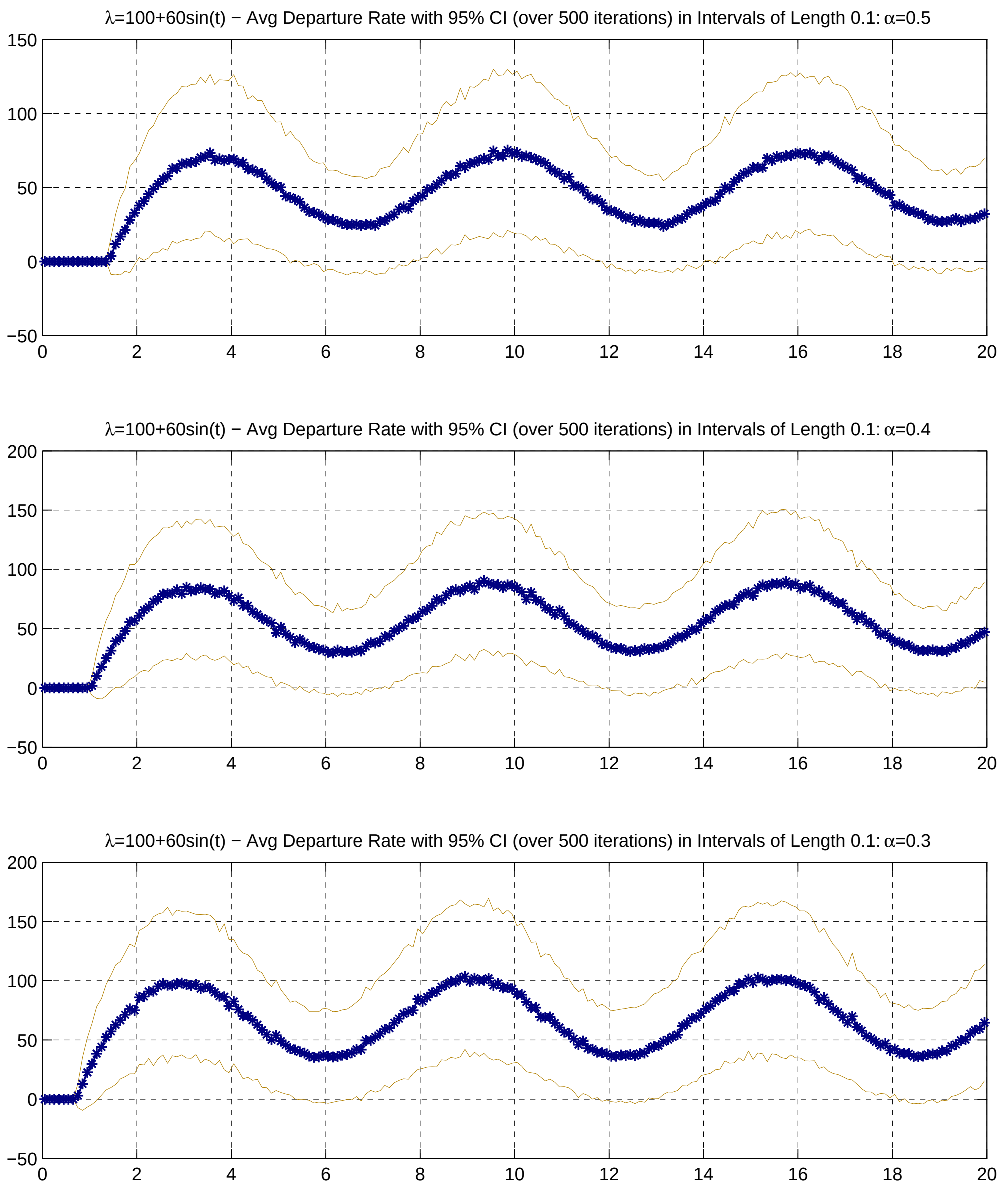


Figure 17: Estimated departure rate function from the $M_t/H_2/s+M$ queue with sinusoidal arrival rate $\lambda(t) = 100 + 60sin(t)$ with low QoS (high abandonment probability) targets, starting out empty: based on 1000 replications using subintervals of length 0.1. 95% confidence intervals bands are shown.



We now apply the KS tests from [1, 2] to the departure process data. In each case we have data from 500 independent replications. We consider various ways to combine the data. The power increases if we use more data in each test, but then we have fewer cases to evaluate. Since the data were rounded, we first un-round the data, but here we show results for both the unrounded and rounded data. As discussed in [1], the rounding makes a big difference.

Table 1: CU and Lewis KS test on $[6, 20]$ with different values of L. Test results over 500 iterations.

				Raw						Unrounded (+ rand/50)					
				L=0.5		L=2		L=14		L=0.5		L=2		L=14	
Type	α	Result	# n	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis
Const	$\alpha = 0.5$	Avg p-val	879.0	0.12	0.00	0.40	0.00	0.30	0.00	0.49	0.50	0.49	0.48	0.30	0.48
		# pass		333	0	446	0	372	0	472	478	472	468	372	476
	$\alpha = 0.4$	Avg p-val	1073.3	0.09	0.00	0.39	0.00	0.28	0.00	0.46	0.49	0.48	0.49	0.28	0.49
		# pass		271	0	454	0	353	0	474	467	469	465	353	472
	$\alpha = 0.3$	Avg p-val	1280.3	0.08	0.00	0.37	0.00	0.26	0.00	0.49	0.50	0.49	0.51	0.27	0.49
		# pass		244	0	456	0	362	0	479	471	472	478	360	484
	$\alpha = 0.02$	Avg p-val	1853.5	0.05	0.00	0.35	0.00	0.48	0.00	0.50	0.51	0.50	0.50	0.49	0.50
		# pass		171	0	449	0	467	0	478	477	478	471	467	474
	$\alpha = 0.01$	Avg p-val	1876.6	0.04	0.00	0.34	0.00	0.51	0.00	0.48	0.50	0.49	0.50	0.51	0.50
		# pass		165	0	445	0	474	0	476	481	476	482	474	482
	$\alpha = 0.005$	Avg p-val	1886.6	0.04	0.00	0.35	0.00	0.48	0.00	0.49	0.51	0.50	0.51	0.49	0.51
		# pass		161	0	444	0	470	0	474	483	473	475	473	480
20sin(t)	$\alpha = 0.5$	Avg p-val	883.9	0.14	0.00	0.39	0.00	0.16	0.00	0.54	0.51	0.47	0.51	0.16	0.46
		# pass		351	0	459	0	310	0	475	476	473	465	308	460
	$\alpha = 0.4$	Avg p-val	1081.4	0.10	0.00	0.40	0.00	0.15	0.00	0.50	0.50	0.51	0.47	0.15	0.44
		# pass		308	0	468	0	281	0	471	467	481	471	291	463
	$\alpha = 0.3$	Avg p-val	1280.7	0.08	0.00	0.37	0.00	0.16	0.00	0.51	0.50	0.49	0.51	0.17	0.48
		# pass		279	0	458	0	295	0	477	480	470	483	296	477
	$\alpha = 0.02$	Avg p-val	1856.3	0.05	0.00	0.32	0.00	0.22	0.00	0.48	0.52	0.46	0.53	0.23	0.51
		# pass		170	0	437	0	395	0	468	483	469	480	402	475
	$\alpha = 0.01$	Avg p-val	1882.1	0.04	0.00	0.33	0.00	0.23	0.00	0.50	0.51	0.47	0.51	0.24	0.50
		# pass		171	0	435	0	399	0	478	475	461	474	404	474
	$\alpha = 0.005$	Avg p-val	1892.4	0.04	0.00	0.32	0.00	0.21	0.00	0.48	0.50	0.46	0.50	0.22	0.48
		# pass		152	0	446	0	363	0	475	467	468	474	377	462
60sin(t)	$\alpha = 0.5$	Avg p-val	893.7	0.13	0.00	0.40	0.00	0.01	0.00	0.51	0.48	0.49	0.45	0.01	0.06
		# pass		357	0	454	0	19	0	484	470	469	464	18	126
	$\alpha = 0.4$	Avg p-val	1090.0	0.10	0.00	0.36	0.00	0.01	0.00	0.49	0.50	0.45	0.43	0.01	0.06
		# pass		302	0	454	0	9	0	472	476	466	450	10	110
	$\alpha = 0.3$	Avg p-val	1285.3	0.08	0.00	0.33	0.00	0.00	0.00	0.49	0.51	0.43	0.41	0.00	0.05
		# pass		273	0	421	0	6	0	464	471	451	450	6	101
	$\alpha = 0.02$	Avg p-val	1874.3	0.05	0.00	0.24	0.00	0.00	0.00	0.51	0.49	0.35	0.38	0.00	0.04
		# pass		177	0	373	0	0	0	477	479	417	418	0	89
	$\alpha = 0.01$	Avg p-val	1899.7	0.05	0.00	0.26	0.00	0.00	0.00	0.49	0.50	0.38	0.39	0.00	0.04
		# pass		172	0	373	0	0	0	469	474	426	432	0	106
	$\alpha = 0.005$	Avg p-val	1909.1	0.05	0.00	0.24	0.00	0.00	0.00	0.50	0.48	0.34	0.38	0.00	0.04
		# pass		181	0	360	0	0	0	479	471	399	423	0	102

Complementing the CU and Lewis KS tests on the departure processes from the $M_t/H_2/s_t + M$ model with the sinusoidal arrival rate function in (5), as done in Table 1 and §4.3 of the main paper, we provide both raw and rounded data in Table 7.

Table 2: CU and Lewis KS test on $[6, 20]$ with different values of L. 100 tests of 5 iterations each.

				Raw						Unrounded (+ rand/50)					
				L=0.5		L=2		L=14		L=0.5		L=2		L=14	
Type	α	Result	# n	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis
Const	$\alpha = 0.5$	Avg p-val	3453.9	0.00	0.00	0.25	0.00	0.31	0.00	0.54	0.53	0.51	0.53	0.33	0.52
		# pass		0	0	88	0	77	0	96	96	98	93	78	96
	$\alpha = 0.4$	Avg p-val	4130.4	0.00	0.00	0.20	0.00	0.24	0.00	0.48	0.46	0.46	0.53	0.26	0.45
		# pass		0	0	75	0	67	0	96	92	91	96	70	93
	$\alpha = 0.3$	Avg p-val	4858.4	0.00	0.00	0.20	0.00	0.24	0.00	0.50	0.50	0.51	0.50	0.26	0.49
		# pass		0	0	80	0	65	0	93	96	97	98	67	100
	$\alpha = 0.02$	Avg p-val	6772.7	0.00	0.00	0.16	0.00	0.40	0.00	0.47	0.46	0.50	0.50	0.44	0.44
		# pass		0	0	71	0	90	0	95	92	96	91	90	93
	$\alpha = 0.01$	Avg p-val	6854.3	0.00	0.00	0.14	0.00	0.44	0.00	0.46	0.47	0.46	0.49	0.48	0.41
		# pass		0	0	71	0	90	0	91	89	96	93	91	91
	$\alpha = 0.005$	Avg p-val	6890.7	0.00	0.00	0.16	0.00	0.42	0.00	0.52	0.41	0.51	0.46	0.46	0.47
		# pass		0	0	72	0	88	0	96	90	94	93	90	92
20sin(t)	$\alpha = 0.5$	Avg p-val	3403.7	0.00	0.00	0.26	0.00	0.03	0.00	0.50	0.53	0.50	0.49	0.03	0.42
		# pass		0	0	84	0	19	0	94	97	98	94	22	84
	$\alpha = 0.4$	Avg p-val	4096.5	0.00	0.00	0.23	0.00	0.02	0.00	0.55	0.52	0.51	0.51	0.02	0.34
		# pass		0	0	86	0	13	0	96	95	94	97	11	82
	$\alpha = 0.3$	Avg p-val	4793.2	0.00	0.00	0.19	0.00	0.02	0.00	0.50	0.49	0.46	0.48	0.02	0.37
		# pass		0	0	77	0	9	0	90	93	94	94	14	87
	$\alpha = 0.02$	Avg p-val	6756.6	0.00	0.00	0.09	0.00	0.01	0.00	0.42	0.47	0.34	0.53	0.02	0.42
		# pass		0	0	49	0	8	0	91	97	80	95	14	91
	$\alpha = 0.01$	Avg p-val	6844.9	0.00	0.00	0.08	0.00	0.01	0.00	0.46	0.48	0.34	0.52	0.02	0.47
		# pass		0	0	52	0	9	0	94	94	86	100	11	93
	$\alpha = 0.005$	Avg p-val	6874.5	0.00	0.00	0.12	0.00	0.01	0.00	0.47	0.48	0.41	0.48	0.01	0.48
		# pass		0	0	55	0	3	0	98	95	91	94	6	95
60sin(t)	$\alpha = 0.5$	Avg p-val	3304.5	0.00	0.00	0.27	0.00	0.00	0.00	0.50	0.51	0.51	0.34	0.00	0.00
		# pass		0	0	82	0	0	0	97	96	93	82	0	0
	$\alpha = 0.4$	Avg p-val	3993.6	0.00	0.00	0.16	0.00	0.00	0.00	0.46	0.48	0.38	0.24	0.00	0.00
		# pass		0	0	65	0	0	0	94	95	88	78	0	0
	$\alpha = 0.3$	Avg p-val	4671.8	0.00	0.00	0.11	0.00	0.00	0.00	0.48	0.45	0.31	0.21	0.00	0.00
		# pass		0	0	48	0	0	0	93	97	74	55	0	0
	$\alpha = 0.02$	Avg p-val	6751.5	0.00	0.00	0.01	0.00	0.00	0.00	0.45	0.45	0.07	0.13	0.00	0.00
		# pass		0	0	8	0	0	0	90	89	35	48	0	0
	$\alpha = 0.01$	Avg p-val	6836.1	0.00	0.00	0.03	0.00	0.00	0.00	0.50	0.55	0.14	0.14	0.00	0.00
		# pass		0	0	14	0	0	0	96	95	44	51	0	0
	$\alpha = 0.005$	Avg p-val	6869.0	0.00	0.00	0.01	0.00	0.00	0.00	0.47	0.47	0.07	0.17	0.00	0.00
		# pass		0	0	7	0	0	0	86	92	34	52	0	0

Table 3: CU and Lewis KS test on $[6, 20]$ with different values of L. 50 tests of 10 iterations each.

				Raw						Unrounded (+ rand/50)					
				L=0.5		L=2		L=14		L=0.5		L=2		L=14	
Type	α	Result	# n	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis
Const	$\alpha = 0.5$	Avg p-val	6907.7	0.00	0.00	0.15	0.00	0.30	0.00	0.52	0.58	0.50	0.56	0.32	0.50
		# pass		0	0	38	0	36	0	48	48	49	49	36	49
	$\alpha = 0.4$	Avg p-val	8260.8	0.00	0.00	0.12	0.00	0.20	0.00	0.53	0.41	0.47	0.47	0.23	0.42
		# pass		0	0	31	0	29	0	47	46	46	48	32	45
	$\alpha = 0.3$	Avg p-val	9716.8	0.00	0.00	0.13	0.00	0.18	0.00	0.50	0.48	0.54	0.53	0.20	0.58
		# pass		0	0	32	0	28	0	44	47	46	49	30	50
	$\alpha = 0.02$	Avg p-val	13545.5	0.00	0.00	0.08	0.00	0.36	0.00	0.54	0.47	0.53	0.41	0.42	0.38
		# pass		0	0	28	0	45	0	48	46	46	46	45	43
	$\alpha = 0.01$	Avg p-val	13708.7	0.00	0.00	0.08	0.00	0.34	0.00	0.46	0.45	0.54	0.46	0.40	0.48
		# pass		0	0	29	0	41	0	47	47	47	45	41	44
	$\alpha = 0.005$	Avg p-val	13781.3	0.00	0.00	0.07	0.00	0.32	0.00	0.52	0.44	0.51	0.41	0.38	0.43
		# pass		0	0	28	0	42	0	47	47	49	45	42	47
20sin(t)	$\alpha = 0.5$	Avg p-val	6807.3	0.00	0.00	0.15	0.00	0.00	0.00	0.47	0.48	0.50	0.52	0.00	0.36
		# pass		0	0	37	0	0	0	48	49	49	50	1	41
	$\alpha = 0.4$	Avg p-val	8192.9	0.00	0.00	0.15	0.00	0.00	0.00	0.52	0.47	0.52	0.46	0.00	0.34
		# pass		0	0	38	0	0	0	46	44	47	48	1	38
	$\alpha = 0.3$	Avg p-val	9586.3	0.00	0.00	0.10	0.00	0.00	0.00	0.56	0.54	0.46	0.55	0.00	0.27
		# pass		0	0	29	0	0	0	47	47	46	46	1	41
	$\alpha = 0.02$	Avg p-val	13513.1	0.00	0.00	0.02	0.00	0.00	0.00	0.46	0.49	0.23	0.46	0.00	0.45
		# pass		0	0	10	0	0	0	42	48	33	47	0	47
	$\alpha = 0.01$	Avg p-val	13689.8	0.00	0.00	0.03	0.00	0.00	0.00	0.46	0.51	0.27	0.56	0.00	0.52
		# pass		0	0	11	0	0	0	47	47	38	50	0	47
	$\alpha = 0.005$	Avg p-val	13749.0	0.00	0.00	0.04	0.00	0.00	0.00	0.50	0.44	0.31	0.46	0.00	0.45
		# pass		0	0	13	0	0	0	49	50	37	45	0	46
60sin(t)	$\alpha = 0.5$	Avg p-val	6609.1	0.00	0.00	0.15	0.00	0.00	0.00	0.54	0.48	0.46	0.31	0.00	0.00
		# pass		0	0	34	0	0	0	48	47	45	37	0	0
	$\alpha = 0.4$	Avg p-val	7987.3	0.00	0.00	0.05	0.00	0.00	0.00	0.42	0.53	0.24	0.14	0.00	0.00
		# pass		0	0	15	0	0	0	44	49	39	25	0	0
	$\alpha = 0.3$	Avg p-val	9343.6	0.00	0.00	0.04	0.00	0.00	0.00	0.47	0.40	0.18	0.12	0.00	0.00
		# pass		0	0	8	0	0	0	46	46	27	16	0	0
	$\alpha = 0.02$	Avg p-val	13503.0	0.00	0.00	0.00	0.00	0.00	0.00	0.41	0.48	0.02	0.05	0.00	0.00
		# pass		0	0	0	0	0	0	42	44	3	10	0	0
	$\alpha = 0.01$	Avg p-val	13672.2	0.00	0.00	0.00	0.00	0.00	0.00	0.47	0.48	0.04	0.02	0.00	0.00
		# pass		0	0	0	0	0	0	46	48	10	6	0	0
	$\alpha = 0.005$	Avg p-val	13738.0	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.54	0.02	0.02	0.00	0.00
		# pass		0	0	0	0	0	0	45	45	6	5	0	0

Table 4: CU and Lewis KS test on $[6, 20]$ with different values of L. 20 tests of 25 iterations each.

				Raw						Unrounded (+ rand/50)					
				L=0.5		L=2		L=14		L=0.5		L=2		L=14	
Type	α	Result	# n	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis
Const	$\alpha = 0.5$	Avg p-val	17269.3	0.00	0.00	0.04	0.00	0.19	0.00	0.49	0.65	0.41	0.59	0.25	0.53
		# pass		0	0	5	0	12	0	19	20	20	20	12	20
	$\alpha = 0.4$	Avg p-val	20652.1	0.00	0.00	0.04	0.00	0.25	0.00	0.49	0.55	0.52	0.43	0.30	0.45
		# pass		0	0	6	0	13	0	19	18	19	19	14	19
	$\alpha = 0.3$	Avg p-val	24292.1	0.00	0.00	0.03	0.00	0.12	0.00	0.45	0.59	0.59	0.47	0.16	0.48
		# pass		0	0	4	0	11	0	16	18	20	19	14	18
	$\alpha = 0.02$	Avg p-val	33863.7	0.00	0.00	0.02	0.00	0.30	0.00	0.43	0.37	0.64	0.34	0.40	0.36
		# pass		0	0	2	0	13	0	20	17	19	17	15	17
	$\alpha = 0.01$	Avg p-val	34271.7	0.00	0.00	0.01	0.00	0.20	0.00	0.42	0.29	0.47	0.30	0.28	0.22
		# pass		0	0	1	0	16	0	16	18	19	16	17	15
	$\alpha = 0.005$	Avg p-val	34453.3	0.00	0.00	0.01	0.00	0.23	0.00	0.56	0.38	0.52	0.35	0.33	0.39
		# pass		0	0	0	0	14	0	19	16	19	16	16	18
20sin(t)	$\alpha = 0.5$	Avg p-val	17018.3	0.00	0.00	0.06	0.00	0.00	0.00	0.53	0.41	0.50	0.44	0.00	0.18
		# pass		0	0	10	0	0	0	20	19	18	19	0	10
	$\alpha = 0.4$	Avg p-val	20482.4	0.00	0.00	0.03	0.00	0.00	0.00	0.55	0.49	0.46	0.61	0.00	0.11
		# pass		0	0	4	0	0	0	20	20	20	19	0	9
	$\alpha = 0.3$	Avg p-val	23965.8	0.00	0.00	0.02	0.00	0.00	0.00	0.51	0.48	0.39	0.60	0.00	0.16
		# pass		0	0	1	0	0	0	19	20	18	18	0	13
	$\alpha = 0.02$	Avg p-val	33782.8	0.00	0.00	0.00	0.00	0.00	0.00	0.43	0.48	0.12	0.47	0.00	0.27
		# pass		0	0	0	0	0	0	18	19	7	16	0	15
	$\alpha = 0.01$	Avg p-val	34224.4	0.00	0.00	0.00	0.00	0.00	0.00	0.36	0.53	0.11	0.36	0.00	0.34
		# pass		0	0	0	0	0	0	17	19	9	18	0	16
	$\alpha = 0.005$	Avg p-val	34372.5	0.00	0.00	0.00	0.00	0.00	0.00	0.44	0.52	0.16	0.53	0.00	0.34
		# pass		0	0	0	0	0	0	19	19	9	18	0	17
60sin(t)	$\alpha = 0.5$	Avg p-val	16522.7	0.00	0.00	0.03	0.00	0.00	0.00	0.52	0.47	0.42	0.05	0.00	0.00
		# pass		0	0	4	0	0	0	19	19	18	7	0	0
	$\alpha = 0.4$	Avg p-val	19968.2	0.00	0.00	0.00	0.00	0.00	0.00	0.27	0.38	0.09	0.02	0.00	0.00
		# pass		0	0	0	0	0	0	18	17	8	2	0	0
	$\alpha = 0.3$	Avg p-val	23358.9	0.00	0.00	0.00	0.00	0.00	0.00	0.33	0.40	0.03	0.01	0.00	0.00
		# pass		0	0	0	0	0	0	18	15	5	0	0	0
	$\alpha = 0.02$	Avg p-val	33757.6	0.00	0.00	0.00	0.00	0.00	0.00	0.42	0.29	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	18	14	0	0	0	0
	$\alpha = 0.01$	Avg p-val	34180.4	0.00	0.00	0.00	0.00	0.00	0.00	0.49	0.55	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	20	20	0	0	0	0
	$\alpha = 0.005$	Avg p-val	34344.9	0.00	0.00	0.00	0.00	0.00	0.00	0.28	0.38	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	14	18	0	0	0	0

Table 5: CU and Lewis KS test on $[6, 20]$ with different values of L. 5 tests of 100 iterations each.

				Raw						Unrounded (+ rand/50)					
				L=0.5		L=2		L=14		L=0.5		L=2		L=14	
Type	α	Result	# n	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis
Const	$\alpha = 0.5$	Avg p-val	69077.2	0.00	0.00	0.00	0.00	0.12	0.00	0.43	0.42	0.72	0.67	0.20	0.37
		# pass		0	0	0	0	2	0	4	5	5	5	2	4
	$\alpha = 0.4$	Avg p-val	82608.2	0.00	0.00	0.00	0.00	0.12	0.00	0.78	0.62	0.52	0.44	0.25	0.27
		# pass		0	0	0	0	3	0	5	4	5	4	3	5
	$\alpha = 0.3$	Avg p-val	97168.4	0.00	0.00	0.00	0.00	0.08	0.00	0.25	0.52	0.60	0.53	0.15	0.47
		# pass		0	0	0	0	1	0	4	4	5	4	1	5
	$\alpha = 0.02$	Avg p-val	135454.8	0.00	0.00	0.00	0.00	0.07	0.00	0.46	0.20	0.56	0.16	0.18	0.16
		# pass		0	0	0	0	2	0	5	2	4	2	2	2
	$\alpha = 0.01$	Avg p-val	137086.8	0.00	0.00	0.00	0.00	0.02	0.00	0.25	0.07	0.46	0.03	0.07	0.04
		# pass		0	0	0	0	1	0	4	2	4	1	2	2
	$\alpha = 0.005$	Avg p-val	137813.0	0.00	0.00	0.00	0.00	0.00	0.00	0.39	0.09	0.48	0.11	0.01	0.06
		# pass		0	0	0	0	0	0	5	2	5	3	0	2
20sin(t)	$\alpha = 0.5$	Avg p-val	68073.2	0.00	0.00	0.00	0.00	0.00	0.00	0.64	0.54	0.68	0.45	0.00	0.00
		# pass		0	0	0	0	0	0	5	5	5	5	0	0
	$\alpha = 0.4$	Avg p-val	81929.4	0.00	0.00	0.00	0.00	0.00	0.00	0.23	0.60	0.18	0.40	0.00	0.00
		# pass		0	0	0	0	0	0	5	5	3	5	0	0
	$\alpha = 0.3$	Avg p-val	95863.0	0.00	0.00	0.00	0.00	0.00	0.00	0.46	0.47	0.16	0.37	0.00	0.00
		# pass		0	0	0	0	0	0	5	5	3	4	0	0
	$\alpha = 0.02$	Avg p-val	135131.2	0.00	0.00	0.00	0.00	0.00	0.00	0.09	0.48	0.00	0.73	0.00	0.11
		# pass		0	0	0	0	0	0	2	4	0	4	0	2
	$\alpha = 0.01$	Avg p-val	136897.6	0.00	0.00	0.00	0.00	0.00	0.00	0.09	0.28	0.00	0.52	0.00	0.24
		# pass		0	0	0	0	0	0	2	4	0	5	0	3
	$\alpha = 0.005$	Avg p-val	137490.0	0.00	0.00	0.00	0.00	0.00	0.00	0.27	0.34	0.02	0.28	0.00	0.11
		# pass		0	0	0	0	0	0	4	3	1	4	0	1
60sin(t)	$\alpha = 0.5$	Avg p-val	66090.8	0.00	0.00	0.00	0.00	0.00	0.00	0.53	0.51	0.33	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	5	4	4	0	0	0
	$\alpha = 0.4$	Avg p-val	79872.8	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.28	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	1	3	0	0	0	0
	$\alpha = 0.3$	Avg p-val	93435.6	0.00	0.00	0.00	0.00	0.00	0.00	0.10	0.22	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	3	4	0	0	0	0
	$\alpha = 0.02$	Avg p-val	135030.4	0.00	0.00	0.00	0.00	0.00	0.00	0.26	0.12	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	2	2	0	0	0	0
	$\alpha = 0.01$	Avg p-val	136721.6	0.00	0.00	0.00	0.00	0.00	0.00	0.25	0.13	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	5	4	0	0	0	0
	$\alpha = 0.005$	Avg p-val	137379.6	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.11	0.00	0.00	0.00	0.00
		# pass		0	0	0	0	0	0	1	2	0	0	0	0

Table 6: CU and Lewis KS test on $[6, 20]$ with different values of L. 1 test of 500 iterations each.

			Raw						Unrounded (+ rand/50)					
			L=0.5		L=2		L=14		L=0.5		L=2		L=14	
Type	α	# n	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis
Const	$\alpha = 0.5$	345386.0	0.00	0.00	0.00	0.00	0.00	0.00	0.68	0.17	0.65	0.56	0.00	0.01
	$\alpha = 0.4$	413041.0	0.00	0.00	0.00	0.00	0.00	0.00	0.85	0.08	0.41	0.03	0.00	0.04
	$\alpha = 0.3$	485842.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.61	0.07	0.00	0.18
	$\alpha = 0.02$	677274.0	0.00	0.00	0.00	0.00	0.00	0.00	0.85	0.00	0.04	0.00	0.00	0.00
	$\alpha = 0.01$	685434.0	0.00	0.00	0.00	0.00	0.00	0.00	0.18	0.00	0.02	0.00	0.00	0.00
	$\alpha = 0.005$	689065.0	0.00	0.00	0.00	0.00	0.00	0.00	0.14	0.00	0.36	0.00	0.00	0.00
20sin(t)	$\alpha = 0.5$	340366.0	0.00	0.00	0.00	0.00	0.00	0.00	0.72	0.06	0.72	0.06	0.00	0.00
	$\alpha = 0.4$	409647.0	0.00	0.00	0.00	0.00	0.00	0.00	0.14	0.72	0.01	0.35	0.00	0.00
	$\alpha = 0.3$	479315.0	0.00	0.00	0.00	0.00	0.00	0.00	0.61	0.09	0.00	0.04	0.00	0.00
	$\alpha = 0.02$	675656.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.00	0.68	0.00	0.00
	$\alpha = 0.01$	684488.0	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.07	0.00	0.86	0.00	0.00
	$\alpha = 0.005$	687450.0	0.00	0.00	0.00	0.00	0.00	0.00	0.09	0.01	0.00	0.29	0.00	0.00
60sin(t)	$\alpha = 0.5$	330454.0	0.00	0.00	0.00	0.00	0.00	0.00	0.22	0.01	0.00	0.00	0.00	0.00
	$\alpha = 0.4$	399364.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.00	0.00	0.00	0.00
	$\alpha = 0.3$	467178.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	$\alpha = 0.02$	675152.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	$\alpha = 0.01$	683608.0	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.01	0.00	0.00	0.00	0.00
	$\alpha = 0.005$	686898.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 7: The CU and Lewis KS tests applied to the departure processes over $[6, 20]$ from the $M_t/H_2/s_t + M$ model with the sinusoidal arrival rate function in (5) in 18 cases: 3 relative amplitudes [$r = 0$ (constant), 0.2 and 0.6] and 6 abandonment probability targets α, 3 low QoS [.5, .4 and .3] and 3 high QoS [.02, .01 and .005]. The KS tests are applied 500 times, once for each replication in 6 cases: with raw and rounded data and three subinterval lengths L: 0.1, 2 and 14.

arrival	aband.		sample	raw						unrounded (+ rand/50)					
rate	prob.		size	L=0.5		L=2		L=14		L=0.5		L=2		L=14	
fct.	target	result	# n	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis	CU	Lewis
$r = 0$	$\alpha = .5$	p-val	879	0.12	0.00	0.40	0.00	0.30	0.00	0.49	0.50	0.49	0.48	0.30	0.48
		# pass		333	0	446	0	372	0	472	478	472	468	372	476
const.	$\alpha = .4$	p-val	1073	0.09	0.00	0.39	0.00	0.28	0.00	0.46	0.49	0.48	0.49	0.28	0.49
		# pass		271	0	454	0	353	0	474	467	469	465	353	472
	$\alpha = .3$	p-val	1280	0.08	0.00	0.37	0.00	0.26	0.00	0.49	0.50	0.49	0.51	0.27	0.49
		# pass		244	0	456	0	362	0	479	471	472	478	360	484
	$\alpha = .02$	p-val	1854	0.05	0.00	0.35	0.00	0.48	0.00	0.50	0.51	0.50	0.50	0.49	0.50
		# pass		171	0	449	0	467	0	478	477	478	471	467	474
	$\alpha = .01$	p-val	1877	0.04	0.00	0.34	0.00	0.51	0.00	0.48	0.50	0.49	0.50	0.51	0.50
		# pass		165	0	445	0	474	0	476	481	476	482	474	482
	$\alpha = .005$	p-val	1887	0.04	0.00	0.35	0.00	0.48	0.00	0.49	0.51	0.50	0.51	0.49	0.51
		# pass		161	0	444	0	470	0	474	483	473	475	473	480
$r = 0.2$	$\alpha = .5$	p-val	884	0.14	0.00	0.39	0.00	0.16	0.00	0.54	0.51	0.47	0.51	0.16	0.46
		# pass		351	0	459	0	310	0	475	476	473	465	308	460
sine	$\alpha = .4$	p-val	1081	0.10	0.00	0.40	0.00	0.15	0.00	0.50	0.50	0.51	0.47	0.15	0.44
		# pass		308	0	468	0	281	0	471	467	481	471	291	463
	$\alpha = .3$	p-val	1281	0.08	0.00	0.37	0.00	0.16	0.00	0.51	0.50	0.49	0.51	0.17	0.48
		# pass		279	0	458	0	295	0	477	480	470	483	296	477
	$\alpha = .02$	p-val	1856	0.05	0.00	0.32	0.00	0.22	0.00	0.48	0.52	0.46	0.53	0.23	0.51
		# pass		170	0	437	0	395	0	468	483	469	480	402	475
	$\alpha = .01$	p-val	1882	0.04	0.00	0.33	0.00	0.23	0.00	0.50	0.51	0.47	0.51	0.24	0.50
		# pass		171	0	435	0	399	0	478	475	461	474	404	474
	$\alpha = .005$	p-val	1892	0.04	0.00	0.32	0.00	0.21	0.00	0.48	0.50	0.46	0.50	0.22	0.48
		# pass		152	0	446	0	363	0	475	467	468	474	377	462
$r = 0.6$	$\alpha = .5$	Avg p-val	894	0.13	0.00	0.40	0.00	0.01	0.00	0.51	0.48	0.49	0.45	0.01	0.06
		# pass		357	0	454	0	19	0	484	470	469	464	18	126
sine	$\alpha = .4$	p-val	1090	0.10	0.00	0.36	0.00	0.01	0.00	0.49	0.50	0.45	0.43	0.01	0.06
		# pass		302	0	454	0	9	0	472	476	466	450	10	110
	$\alpha = .3$	p-val	1285	0.08	0.00	0.33	0.00	0.00	0.00	0.49	0.51	0.43	0.41	0.00	0.05
		# pass		273	0	421	0	6	0	464	471	451	450	6	101
	$\alpha = .02$	p-val	1874	0.05	0.00	0.24	0.00	0.00	0.00	0.51	0.49	0.35	0.38	0.00	0.04
		# pass		177	0	373	0	0	0	477	479	417	418	0	89
	$\alpha = .01$	p-val	1900	0.05	0.00	0.26	0.00	0.00	0.00	0.49	0.50	0.38	0.39	0.00	0.04
		# pass		172	0	373	0	0	0	469	474	426	432	0	106
	$\alpha = .005$	p-val	1909	0.05	0.00	0.24	0.00	0.00	0.00	0.50	0.48	0.34	0.38	0.00	0.04
		# pass		181	0	360	0	0	0	479	471	399	423	0	102

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