Assignment 4: DTMC: state classification and gambler’s ruin problem
- Due on September 18


Other Problems:

1. (Gambler’s Ruin) Consider the Gambler’s ruin DTMC \{X_n, n \geq 0\} on \mathcal{S} = \{0, 1, \ldots, N\}. \ P_{0,0} = 1 = P_{N,N}, while otherwise \ P_{i,i+1} = p, \ P_{i,i-1} = 1 - p, 1 < i < N. Suppose that \ X_0 = i, for 1 < i < N. Show that \ X_n converges to some limiting random variable \ X^{(i)}_\infty almost surely and find its PMF: \ p_k = \mathbb{P}(X^{(i)}_\infty = k), k \in \mathcal{S}.

2. (State-dependent Gambler’s Ruin) Consider the gambler’s ruin problem when \ N = 3, except now we allow the probability \ p (of winning a dollar) to depend on the present state. Whenever \ X_n = 1, the probability that the gambler wins a dollar is \ p_1 (and 1 - p_1 for losing a dollar). Similarly, Whenever \ X_n = 2, the probability that the gambler wins a dollar is \ p_2 (and 1 - p_2 for losing a dollar). Find \ P_1 and \ P_2, the probabilities that that gambler’s fortune reaches 3 before getting ruined, starting with 1 and 2 respectively. Compute for the case when \ p_1 = 0.7 and \ p_2 = 0.3.

3. Consider an ergodic (i.e., irreducible, positive recurrent, and aperiodic) DTMC \{X_n, n \geq 0\} with a finite state space \mathcal{S} = \{1, 2, \ldots, N\} and a transition probability matrix \ P \in \mathbb{R}^{N \times N}. Consider a row vector \ \pi = (\pi_1, \ldots, \pi_N) satisfying the following balance equation:

\[
\pi P = \pi \\
\pi_1 + \cdots + \pi_N = 1.
\]

Note that there are \ N + 1 equations (note \ \pi P = \pi has \ N equations) but only \ N unknowns. Show that the last equation of \ \pi P = \pi:

\[
\sum_{i=1}^{N} \pi_i P_{i,N} = \pi_N \tag{1}
\]

is redundant, i.e., show that (1) can be derived from the other \ N equations.

Remark: In fact, any equation, not necessarily the last one, in \ \pi P = \pi is redundant.

Reading: Read Green Ross Sections 4.4, 4.5.1, 4.6.