Assignment 4: DTMC: state classification and gambler's ruin problem

- Due on September 18

From Green Ross: Do Exercise 4.14, 4.17, 4.57, 4.59 11 ed. (4.60 10 ed.) in Chapter 4.

Other Problems:

1. **(Gambler's Ruin)** Consider the Gambler's ruin DTMC $\{X_n, n \geq 0\}$ on $\mathcal{S} = \{0, 1, \ldots, N\}$. $P_{0,0} = 1 = P_{N,N}$, while otherwise $P_{i,i+1} = p$, $P_{i,i-1} = 1 - p$, $1 < i < N$. Suppose that $X_0 = i$, for $1 < i < N$. Show that X_n converges to some limiting random variable $X_\infty^{(i)}$ almost surely and find its PMF: $p_k = \mathbb{P}(X_\infty^{(i)} = k)$, $k \in \mathcal{S}$.

2. **(State-dependent Gambler's Ruin)** Consider the gambler's ruin problem when $N = 3$, except now we allow the probability p (of winning a dollar) to depend on the present state. Whenever $X_n = 1$, the probability that the gambler wins a dollar is p_1 (and $1 - p_1$ for losing a dollar). Similarly, Whenever $X_n = 2$, the probability that the gambler wins a dollar is p_2 (and $1 - p_2$ for losing a dollar). Find P_1 and P_2, the probabilities that that gambler's fortune reaches 3 before getting ruined, starting with 1 and 2 respectively. Compute for the case when $p_1 = 0.7$ and $p_2 = 0.3$.

3. Consider an ergodic (i.e., irreducible, positive recurrent, and aperiodic) DTMC $\{X_n, n \geq 0\}$ with a finite state space $\mathcal{S} = \{1, 2, \ldots, N\}$ and a transition probability matrix $P \in \mathbb{R}^{N \times N}$. Consider a row vector $\pi \equiv (\pi_1, \ldots, \pi_N)$ satisfying the following balance equation:

$$\begin{aligned} \pi P &= \pi \\ \pi_1 + \cdots + \pi_N &= 1. \end{aligned}$$

   Note that there are $N + 1$ equations (note $\pi P = \pi$ has N equations) but only N unknowns. Show that the last equation of $\pi P = \pi$:

$$\sum_{i=1}^{N} \pi_i P_{i,N} = \pi_N \tag{1}$$

   is redundant, i.e., show that (1) can be derived from the other N equations.
   Remark: In fact, any equation, not necessarily the last one, in $\pi P = \pi$ is redundant.

Reading: Read Green Ross Sections 4.4, 4.5.1, 4.6.