Assignment 5:
Steady States of Irreducible DTMCs and Canonical Forms of Reducible DTMCs

- Due on September 25

Do the following exercise in Chapter 4:

Problem 4.19, 4.20, 4.46, 4.49.

Other Problems:

1. Let π_j, $j \in \mathcal{S}$, be the stationary probabilities for a DTMC.

   (a) What is the interpretation of $\pi_i P_{ij}$? Prove your claim.

   (b) Let $A \subset \mathcal{S}$ be a set of states and A^c be the remaining states. Give an interpretation of the following two quantities and prove your claim.

   $$\sum_{i \in A} \sum_{j \in A^c} \pi_i \, P_{ij} \quad \text{and} \quad \sum_{i \in A^c} \sum_{j \in A} \pi_i \, P_{ij}?$$

   (c) Let $N_n(A, A^c)$ be the total number of transitions made from a state in A to one in A^c among the first n transitions; similarly, let $N_n(A^c, A)$ be the total number of transitions made from a state in A^c to one in A among the first n transitions. Prove or disprove the following statement:

   $$|N_n(A, A^c) - N_n(A^c, A)| \leq 1.$$

   (d) Do you think the following equality hold? If yes, prove it; otherwise, give a counterexample. (Hint: Apply your result in (c).)

   $$\sum_{j \in A^c} \sum_{i \in A} \pi_i \, P_{ij} = \sum_{j \in A^c} \sum_{i \in A} \pi_j \, P_{ji}.$$

2. **Gamber's Ruin revisited**

   Consider the gambler's ruin problem with $p = 0.3$ and $N = 5$. Starting with 3 units, determine

   (a) the probability that the gambler's fortune reaches N before 0 using two approaches:

      (i) the Gambler's ruin formula;

      (ii) the fundamental matrix.

   (b) the expected number of steps until the end of the game using two approaches:

      (i) the formula in Ross 4.60 (See HW4 solution for the right formula);

      (ii) the fundamental matrix.

   (c) the expected amount of time the gambler has 2 units,

   (d) the probability that the gambler ever has a fortune of 1 (i.e., the probability that state 1 is ever visited).

3. **Lord Voldemort (LV) hunting Harry Potter (HP)**
LV teleports between two locations: 1 and 2 according to a DTMC with transition probability matrix

$$\mathbf{P}^{\text{LV}} = \begin{matrix}(1)\\(2)\end{matrix}\begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}.$$

HP moves between 1 and 2 according to a DTMC with

$$\mathbf{P}^{\text{HP}} = \begin{matrix}(1)\\(2)\end{matrix}\begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}.$$

The hunt ends whenever they meet in the same location. (we all know that HP will win!) Suppose LV starts in location 1 and HP starts in 2 on day 0.

(a) Show that the progress of the hunt, except for knowing the location where it ends, can be described by a 3-state DTMC with one absorbing state and two transient states. Obtain the transition probability matrix for this DTMC.

(b) Find the probability that the hunt ends in 2 days.

(c) What is the average duration of the hunt?

4. **General Canonical Form of A Reducible DTMC**
Consider a DTMC with $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$\mathbf{P} = \begin{matrix}(1)\\(2)\\(3)\\(4)\\(5)\end{matrix}\begin{bmatrix} 0.1 & 0 & 0 & 0.9 & 0 \\ 0 & 0.4 & 0 & 0 & 0.6 \\ 0.3 & 0.3 & 0 & 0.4 & 0 \\ 0.3 & 0 & 0 & 0.7 & 0 \\ 0 & 0.7 & 0 & 0 & 0.3 \end{bmatrix}.$$

(a) Classify the DTMC and put it in the canonical form.

(b) Do you think the following limits exist? Find these limits if yes.

$$\text{(i)} \lim_{n\to\infty} P^n_{1,4}; \quad \text{(ii)} \lim_{n\to\infty} P^n_{1,2}; \quad \text{(iii)} \lim_{n\to\infty} P^n_{3,1}; \quad \text{(iv)} \lim_{n\to\infty} P^n_{3,2}.$$

Reading: Read Green Ross Sections 4.4, 4.5.1, 4.6, 4.8. If you have time and want to do extra exercises, I recommend 4.27, 4.32, 4.41, 4.47 (they have answers in the back).