Assignment 7: *Due on October 11*

Properties of Exponential and Definitions of Poisson Processes

From Green Ross: Do Exercise 5.4 and 5.45 in Chapter 5.

Other Problems:

0. **Strong Lack of Memory Property (SLOMP)**
Consider 3 nonnegative r.v.'s X, Y and Z, where $Z \sim \text{Exp}(\lambda)$, $X \perp\!\!\!\perp Z$, $Y \perp\!\!\!\perp Z$. Show that
$$\mathbb{P}(Z > X + Y | Z > X) = \mathbb{P}(Z > Y).$$

1. For Poisson processes, show that Definition 1 implies Definition 2.

2. Consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ, for $s < t$
   (a) Find the conditional distribution of $N(s)$ given $N(t) = n$.
   (b) Compute $Cov(N(s), N(t))$.

   Hint: Use the property of stationary and independent increment.

3. **Compound Poisson Process (CPP)**
Definition: Consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ, that is independent of an I.I.D. sequence $X_1, X_2, \ldots$ with mean $\mathbb{E}[X] = \mu$ and variance $Var(X) = \sigma^2$. Let $Y(t) \equiv \sum_{i=1}^{N(t)} X_i$, then this new continuous-time stochastic process $\{Y(t), t \geq 0\}$ is called *compound Poisson Process* (CPP).

   Remark: One may observe the limitation of PP due to its unit jump size. In real life, there are many cases in which you have to consider non-unit jump sizes, e.g., people arrive at restaurants in groups. Now, the random increment X_i can characterize the desired jump size. Note that a CPP $\{Y(t), t \geq 0\}$ is NOT necessarily a counting process if you intentionally make $\mathbb{P}(X_i < 0) > 0$ (the sample path may decrease in such cases since the realization of X_i's may be negative). Here are some concrete examples for CPP:

   (i) If $X_i = 1$, then $Y(t) = N(t)$, i.e., CPP degenerates to PP.
   (ii) Let X_i be the number of customers on bus i and let buses arrival $\sim PP(\lambda)$, then $Y(t)$ counts the total number of customer arrivals by t.
   (iii) Let X_i be the amount of cash withdrawal of the ith customer and let customer arrival $\sim PP(\lambda)$, then $Y(t)$ denotes the total amount of cash withdrawn by t.

   Answer the following questions:

   (a) Does $\{Y(t), t \geq 0\}$ have *independent* and/or *stationary* increments? Prove you claim.
   (b) Find the mean $\mathbb{E}[Y(t)]$, variance $Var(Y(t))$, and coviance $Cov(Y(s), Y(t))$, for $0 \leq s < t$.
   (c) Find $Cov(N(t), Y(t))$.

   Hint: Condition on $N(t)$ for $(b) - (c)$. Find $\mathbb{E}[Y(t)^2]$ first to compute the variance in (b).

Reading: Read Green Ross Sections 5.1-5.3 and Exercise 5.18 which has answer in the back.