Assignment 8: *Due on October 18*

Poisson Processes: sampling, superposition and conditional arrival times

From Green Ross: Do Exercise 5.42, 5.52 in Chapter 5.

Extra part for 5.52: Find the expected duration of the game.

Other Problems:

1. The number of trials to be performed is a Poisson random variable with mean λ. Each trial has n possible outcomes and, independent of everything else, results in outcome number i with probability P_i, $\sum_{i=1}^{n} P_i = 1$. Let X_j denote the number of outcomes that occur exactly j times, $j = 0, 1, \ldots$ Compute $\mathbb{E}[X_j]$ and $Var(X_j)$.

2. Shocks occur according to a $PP(\lambda)$. Suppose that each shock, independently, causes the system to fail with probability p. Let N denote the number of shocks that it takes for the system to fail and let T denote the time of failure. Find the conditional probability $\mathbb{P}(N = n|T = t)$. What kind of distribution does the conditional r.v. $(N|T = t)$ follow?

Reading: Read Green Ross Sections 5.3.

Hints for Assignment 8

OP 1: Let Y_n be the number of customers arriving during the service period of the $(n+1)$th customer. Establish a recursion for X_n.

OP 2: Let X be the time until the next car arrival. Condition on event $\{X > T\}$ and $\{X \leq T\}$.

OP 3: (a) Recall that min of Exp's is again Exp and apply the lack-of-memory property.

(b) We have already checked Properties (i)-(iii) in class, it remains to check (iv).

Remark: you may as well consider how to prove **superposition** *using definition 2 and* **thinning** *using definition 1.*

OP 4: Consider a $PP(\lambda)$ and we count in $[0,1]$. Sample the PP to n independent PP's $\{N_1(t)\}, \ldots, \{N_n(t)\}$ to count the number of outcome i in $[0,1]$, $i = 1, \ldots, n$. Now it's time to consider our best friend: indicator r.v.'s. Let $I_{i,j} \equiv \mathbf{1}_{\{\text{outcome } i \text{ occurs } j \text{ times}\}}$.

First, the desired r.v. $X_j = \sum_{i=1}^n I_{i,j}$. Second, it is not hard to see that $\mathbb{P}(I_{i,j} = 1) = \mathbb{P}(N_i(1) = j)$, and the independence of $I_{1,j}, I_{2,j}, \ldots, I_{n,j}$ is obviously followed. Third, you know what to do...

OP 5: Hint 1: First, note that $\mathbb{P}(N = n | T = t) = \lim_{h \to 0} \mathbb{P}(N = n | t < T < t + h)$. Also,

$$\mathbb{P}(N = n | t < T < t + h) = \frac{\mathbb{P}(N = n, t < T < t + h)}{\mathbb{P}(t < T < t + h)} = \frac{\mathbb{P}(t < T < t + h | N = n)\, \mathbb{P}(N = n)}{\mathbb{P}(t < T < t + h)}.$$

Hint 2: Note that $T = S_N$ and $(T | N = n) = S_n$, which implies that

$$\mathbb{P}(N = n | t < T < t + h) \approx \frac{\left(f_{T|N=n}(t) \cdot h\right)\, \mathbb{P}(N = n)}{f_T(t) \cdot h} = \frac{f_{S_n}(t)\, \mathbb{P}(N = n)}{f_{S_N}(t)}.$$

Hint 3: It remains to find the PDF's for S_n and S_N. We know that $S_n \sim Erlang(n, \lambda)$. To determine the distribution of S_N, note that if we sample the original $PP(\lambda)$ w.p. p, we can obtain a new process $\{N_f(t), t \geq 0\} \sim PP(\lambda p)$, which counts the number of system failures by time t. It is not hard to see that T is the interarrival time (or the occurrence time of the first event) of this new $PP(\lambda p)$. Then you should know what kind of distribution T follows. Alternatively, since you can write $T = S_N = \sum_{i=1}^N X_i$, Extra Problem $1(b)(ii)$ should help at this point.