Assignment 9: *Due on October 25*

Poisson Processes: $M/GI/\infty$ queues and NHPP

From Green Ross: Do Exercise 5.63, 5.70, 5.78, 5.80, 5.81, 5.83 in Chapter 5.

Notation change for 5.81: It seems that 5.81 repeats notation: the distribution F in (b) coincides with the F in (a). So Let's change the F to G in (b) (We've been using G for the service CDF in infinite-server queues).

Extra part for 5.80: (d) Find the distribution of T_2.

Extra part for 5.83: (b) Suppose that $\{N(t), t \geq 0\}$ is an NHPP with rate function $\lambda(t)$ and mean function $m(t) \equiv \int_0^t \lambda(s)ds$. Let $m^{-1}(u)$ be the inverse function of $m(t)$, i.e., $m^{-1}(m(t)) = m(m^{-1}(t)) = t$. Let $N^*(t) \equiv N(m^{-1}(t))$. Show that $\{N^*(t), t \geq 0\}$ is a PP with rate $\lambda = 1$.

Remark: Connecting this extra part with the original 5.83, we see that we can convert a PP to an NHPP (and vice versa) with nonlinear time transformation.

Other Problems:

1. (**Equivalence of Two Definitions for NHPP**)
   Show that the **first** definition of NHPP (*Definition 4* in the notes) implies the **second** definition of NHPP (*Definition 5*).

Reading: Read Green Ross Sections 5.3 and Exercise 5.36, 5.60 and 5.79 which have answers in the back.

Hints for Assignment 9

5.63: The hint is given in the book below 5.63.

5.70: Let the arrival time of the first customer be time 0 and let his (her) service time be T. During this customer's service time T, you want no other departures (but there can be arrivals). Conditioning on $\{T = t\}$, consider an $M/GI/\infty$ and let $D(t)$ counts the number of departures, you need $\mathbb{P}(D(t) = 0)$. Let E be the desired event, then

$$\mathbb{P}(E) = \int_0^\infty \mathbb{P}(E|T = t)\, f_T(t)\, dt = \int_0^\infty \mathbb{P}(D(t) = 0)\, f_T(t)\, dt = \dots.$$

5.78: Easy! No hint needed.

5.80: In (d), condition on the value of T_1. What did we do to get the interarrival times for PP?

5.81: In (a), mimic the way we established the conditional distribution of arrival times for PP in class (see notes).

In (b), this is an application of the $M_t/GI/\infty$ queue, where we let the out-of-work times be the service times that are I.I.D. r.v.'s following the CDF G. You can get (b) directly using (a) (as the problem asked you to). conditioning on $N(t) = n$, you have n I.I.D. arrival times (before ordering) following the distribution in (a). It is time to invite our best friend: indicator r.v.'s to construct the number in the system at t

$$\begin{aligned}(X(t)|N(t) = n) &= \sum_{i=1}^n \mathbf{1}(\text{an (unordered) injured worker } i \text{ is out of work at } t) \\ &= \sum_{i=1}^n \mathbf{1}(\text{an (unordered) customer } i \text{ is still in the } M_t/GI/\infty \text{ queue at } t).\end{aligned}$$

Taking expectation of the above expression yields

$$\mathbb{E}[X(t)|N(t) = n] = n\,\mathbb{P}(\text{an (unordered) injured worker is out of work at } t) \equiv n\,\mathbb{P}(E).$$

Consider an arbitrary event in $[0, t]$. Then conditioning on the arrival time $X = s$, $0 < s < t$, this customer will be in the system by t w.p. $P(X > t - s) = 1 - G(t - s)$. So

$$\mathbb{P}(E) = \int_0^t \mathbb{P}(E|X = s) f_X(s)\, ds = \int_0^t \bar{G}(t - s) f_X(s)\, ds.$$

It thus remains to plug in $f_X(s)$ as in (a) and uncondition on $N(t) = n \dots\dots$

Alternatively, you can apply the theory of $M_t/GI/\infty$ introduced in class. Check if your answers coincide.

5.83: Use either *Definition 4* or *5*. (Try both if you can.)

OP 1: Mimic our treatment for PP (Definitions 1 and 2).