Assignment 10: CTMC: Modeling, Birth-and-Death Processes

Due on November 6

From Green Ross: Do Exercises 5.94, 6.9 in Chapters 5 and 6.

Extra part for 5.94: (c) Let \( R_i, i \geq 1 \) denote the distance from an arbitrary point to the \( i \)th closest event to it. Show that, with \( R_0 = 0 \), \( \pi R_i^2 - \pi R_{i-1}^2, i \geq 1 \) are independent exponential random variables each with rate \( \lambda \).

Other Problems:

Definition: (Birth and Death Processes)
A birth-and-death (BAD) process is a CTMC with states \( \{0, 1, \ldots\} \) for which transitions from state \( n \) may go only to either state \( n - 1 \) or state \( n + 1 \). The rate from state \( n \) to state \( n + 1 \) is called birth rate at state \( n \), denoted as \( \lambda_n \); the rate from state \( n \) to state \( n - 1 \) is called death rate at state \( n \), denoted as \( \mu_n \). A BAD is called pure birth process (PBP) if \( \mu_n = 0 \) for all \( n \); a BAD process is called pure death process (PDP) if \( \lambda_n = 0 \) for all \( n \).

Remark: For comparison purpose, think about the gambler’s ruin problem or the simple random walk as discrete-time analogs.

1. (BAD) Consider a BAD process with birth rates \( \lambda_n \) and death rates \( \mu_n \). Stating from \( i \), find the probability that the first \( k \) events are all births.

2. (A Markovian Population Model)
Suppose that a one-celled organism can be in one of two states - either \( A \) or \( B \). An individual in state \( A \) will change to state \( B \) in an exponentially distributed time with rate \( \alpha \); an individual in state \( B \) divides into two new individuals of type \( A \) in an exponentially distributed time with rate \( \beta \). Define an appropriate CTMC for a population of such organisms and determine the appropriate parameters for this model.

3. (Uniformization)
We know a CTMC can be modeled by a third approach: Uniformization. There are 3 steps:
   (i) Choose a PP with a large rate \( \lambda \geq \gamma_i = -Q_{i,i} \), for all \( i \in S \), to generate potential transitions including real transitions and fictitious transitions (from \( i \) back to \( i \)).
   (ii) Apply sampling/thinning the PP: each of the potential transitions is a real transition w.p. \( p_1 = \gamma_i / \lambda \); and a fictitious transition w.p. \( p_2 = 1 - \gamma_i / \lambda \).
   (iii) Modify TPM of embedded DTMC to incorporate all potential transitions
   \[
   \tilde{P}_{i,j} = \frac{Q_{i,j}}{\lambda}, \quad i \neq j \quad \text{and} \quad \tilde{P}_{i,i} = 1 - \sum_{j \geq i} \tilde{P}_{i,j} = 1 - \frac{\sum_{j \in S, j \neq i} Q_{i,j}}{\lambda} = 1 + \frac{Q_{i,i}}{\lambda}.
   \]
   In matrix notation: \( \tilde{P} = I + \frac{1}{\lambda}Q \).

Prove the following results:
(a) The $t$-time transition probability
\[ P_{i,j}(t) = \sum_{k=0}^{\infty} \tilde{p}_{i,j}^{(k)} e^{-\lambda t} \frac{(\lambda t)^k}{k!}. \]  

(1)

(b) Using (1) to show that
\[ P'_{i,j}(0) = Q_{i,j} \quad \text{and} \quad P'_{i,i}(0) = Q_{i,i} \quad \text{for} \quad i \neq j, \quad i,j \in S. \]
Hints for Assignment 10

5.94: This is a generalization for PP: two-dimensional PP (2DPP).

6.9: Easy! Consider each death as an event and connect to $PP(\mu)$.

OP 1: Easy! Use the lack-of-memory property of exponential distribution.

OP 2: Easy! Note that only keep track of the population of one type is not enough.