Assignment 10: CTMC: Modeling, Birth-and-Death Processes

Due on November 6

From Green Ross: Do Exercises 5.94, 6.9 in Chapters 5 and 6.

Extra part for 5.94 : (c) Let R_i, $i \geq 1$ denote the distance from an arbitrary point to the ith closest event to it. Show that, with $R_0 = 0$, $\pi R_i^2 - \pi R_{i-1}^2$, $i \geq 1$ are independent exponential random variables each with rate λ.

Other Problems:

Definition: (Birth and Death Processes)
A *birth-and-death* (BAD) process is a CTMC with states $\{0, 1, \ldots\}$ for which transitions from state n may go only to either state $n-1$ or state $n+1$. The rate from state n to state $n+1$ is called *birth rate* at state n, denoted as λ_n; the rate from state n to state $n-1$ is called *death rate* at state n, denoted as μ_n. A BAD is called *pure birth process* (PBP) if $\mu_n = 0$ for all n; a BAD process is called *pure death process* (PDP) if $\lambda_n = 0$ for all n.

Remark: For comparison purpose, think about the *gambler's ruin* problem or the *simple random walk* as discrete-time analogs.

1. (**BAD**) Consider a BAD process with birth rates λ_n and death rates μ_n. Stating from i, find the probability that the first k events are all births.

2. (**A Markovian Population Model**)
   Suppose that a one-celled organism can be in one of two states - either A or B. An individual in state A will change to state B in an *exponentially* distributed time with rate α; an individual in state B divides into two new individuals of type A in an *exponentially* distributed time with rate β. Define an appropriate CTMC for a population of such organisms and determine the appropriate parameters for this model.

3. (**Uniformization**)
   We know a CTMC can be modeled by a third approach: Uniformization. There are 3 steps:

   (i) Choose a PP with a large rate $\lambda \geq \gamma_i = -Q_{i,i}$, for all $i \in \mathcal{S}$, to generate *potential transitions* including *real transitions* and *fictitious transitions* (from i back to i).

   (ii) Apply sampling/thinning the PP: each of the potential transitions is a *real transition* w.p. $p_1 = \gamma_i/\lambda$; and a *fictitious transition* w.p. $p_2 = 1 - \gamma_i/\lambda$.

   (iii) Modify TPM of embedded DTMC to incorporate all potential transitions

   $$\tilde{P}_{i,j} = \frac{Q_{i,j}}{\lambda}, \quad i \neq j \quad \text{and} \quad \tilde{P}_{i,i} = 1 - \sum_{j, j \geq i} \tilde{P}_{i,j} = 1 - \frac{\sum_{j \in \mathcal{S}, j \neq i} Q_{i,j}}{\lambda} = 1 + \frac{Q_{i,i}}{\lambda}.$$

   In matrix notation: $\tilde{\mathbf{P}} = \mathbf{I} + \frac{1}{\lambda}\mathbf{Q}$.

   Prove the following results:

(a) The t-time transition probability

$$P_{i,j}(t) = \sum_{k=0}^{\infty} \tilde{P}_{i,j}^{(k)} \, e^{-\lambda t} \frac{(\lambda t)^k}{k!}. \tag{1}$$

(b) Using (1) to show that

$$P'_{i,j}(0) = Q_{i,j} \qquad \text{and} \qquad P'_{i,i}(0) = Q_{i,i} \qquad \text{for} \quad i \neq j, \quad i, j \in \mathcal{S}.$$

Reading: Read Green Ross Sections 6.1-6.5 and Exercise 6.2, 6.4, 6.7 and 6.11 which have answers in the back.

Hints for Assignment 10

5.94: This is a generalization for PP: *two-dimensional* PP (2DPP).

6.9: Easy! Consider each death as an event and connect to $PP(\mu)$.

OP 1: Easy! Use the *lack-of-memory* property of *exponential* distribution.

OP 2: Easy! Note that only keep track of the population of one type is not enough.