Assignment 11: *Due on November 13*

CTMC: Steady State Distribution

From Green Ross: Do Exercise 6.12, 6.13 in Chapter 6.

Other Problems:

1. (**The barbershop problem: brutal force approach**)
   Consider the two-server barbershop discussed in problem 2 of ICPS-12 (the $M/M/1 \to \cdot/M/1$ tandem model). Let A_2 be the interarrival time of the second barber. Prove that in steady state, the arrival process to the second barber is a Poisson process with rate λ by show that $A \sim \text{Exp}(\lambda)$. Prove this claim by showing the following two equalities hold:

   (i) The CCDF
   $$\mathbb{P}(A_2 > x) = e^{-\lambda x}, \qquad x > 0.$$

   (ii) The moment-generating function (MGF)
   $$M_{A_2}(\theta) = \mathbb{E}[e^{\theta A_2}] = \frac{\lambda}{\lambda - \theta}, \qquad 0 < \theta < \lambda.$$

2. (**Interpretations on Stationary Distribution**)
   Let α_j, $j \in \mathcal{S}$, be the stationary distribution for a CTMC.

   (a) What is the interpretation of $-\alpha_i Q_{i,i}$ (just guess with your intuition)?

   (b) Prove your claim in (a) (verify your guess with rigorous math).

   (c) What is the interpretation of $\alpha_i Q_{i,j}$ (just guess with your intuition)?

   (d) Prove your claim in (c) (verify your guess with rigorous math).

   Remark: Relate this problem to HW5 Extra Problem 1. It is always good to compare with DTMC while learning CTMC.

3. (**Markov-Modulated Poisson Processes**)
   Consider a $\{N(t), t \geq 0\} \sim PP(\lambda)$.

   (a) Does the limit $\lim_{t\to\infty} N(t)/t$ exist? Intuitively, what do you think should the limit be? In which mode does the above convergence hold?

   (b) Prove your statement in (a).

   Suppose a system can be modeled as a two-state CTMC with a state space $\mathcal{S} \equiv \{1, 2\}$ and transition rate $Q_{1,2} = \gamma_1$ and $Q_{2,1} = \gamma_2$. (This is a simple 2-state CTMC, what's the point?) Ok, what's new is the following: When the system is in state i, events occur in accordance with a Poisson process with rate λ_i, $i = 1, 2$. Let $N(t)$ count the number of events in $[0, t]$.

   (c) Does the limit $\lim_{t\to\infty} N(t)/t$ exist? If yes, find that limit and specify the mode of the convergence; if no, explain why.

   (d) If the initial state of the CTMC is 1, find $\mathbb{E}[N(t)]$.

Remarks:

a. A Markov-Modulated Poisson Process (MMPP) is basically a PP with a rate that changes with time in a Markovian manner.

b. Note that MMPP is different with NHPP or MPP. The rate of MMPP changes with time (randomly) in a Markovian manner; the rate of NHPP changes in a pattern which is deterministic (or say pre-determined); the rate of MPP (see HW11 Extra part for 5.86) is random but not changing with time.

c. MMPP's can be quite useful. Think about the two (or more) states representing "low" and "high" demand, in state 1, the arrival rate is low; in state 2, the rate is high. Just google MMPP and you will find interesting applications.

4. (**Copier Breakdown and Repair**)
Two copier machines in the ISE department are maintained by a single repairman, Justin. Machine i functions for an *exponentially* distributed amount of time with rate γ_i, before it breaks down. The repair times for copier i are *exponential* with rate β_i. But Justin can only work on one machine at a time. Assume that machines are repaired in the order in which they fail. Suppose we wish to construct a CTMC model, with the goal of finding the long-run proportions of time that each copier is working and the repairman is busy. How can we proceed?

   (a) Let $X(t)$ represents the number of working machines at time t. Is $\{X(t), t \geq 0\}$ a CTMC?

   (b) Formulate a CTMC describing the evolution of the system.

   (c) Suppose that $\gamma_1 = 1$, $\beta_1 = 2$, $\gamma_2 = 3$, and $\beta_2 = 4$. Find the stationary distribution.

   (d) Now suppose, instead, that machine 1 is much more important than machine 2, so that the repairman will always work on machine 1 if it is down, regardless of the state of machine 2. Formulate a CTMC for this modified problem and find the stationary distribution.

Problems on Numerical Computation:

5. **Pooh Bear in continuous time**
Consider the Pooh bear example (ICPS-11, Prob.2). Let $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ be the transition rate matrix (TRM). Use "hour" as the time unit here.

   (a) Use Matlab (or any other language) to numerically compute (NOT by hand) $\alpha \equiv (\alpha_A, \alpha_B, \alpha_C)$ using all three modeling approaches:

      i. Modeling approach I: solve the balance equation $\boldsymbol{\alpha}\mathbf{Q} = \boldsymbol{\alpha}$ and $\sum_i \alpha_i = 1$ following the approach described in the Appendix here.

      ii. Modeling approach II: refer to the appendix of HW06 to solve the balance equation for the embedded DTMC with TPM $\mathbf{P}^*$.

      iii. Modeling approach III: refer to the appendix of HW06 to solve the balance equation for the embedded DTMC with TPM $\tilde{\mathbf{P}}$. Try to use different values for $\lambda \geq 1/3$ to confirm that you can obtain the same steady state $\boldsymbol{\alpha}$.

   (b) Let $\mathbf{P}(t) \equiv e^{\mathbf{Q}t}$ be the time-dependent transition probability matrix. Compute $\mathbf{P}(t)$ for $t = 5$ and 50. What do you observe?
   Hint: In MatLab, use the function "expm($\mathbf{A}$)" to compute the matrix exponential for

matrix $\mathbf{A}$. Alternatively, you may use a finite sum to approximate the infinite sum, namely, $e^{\mathbf{A}} = \sum_{n=0}^{\infty} \mathbf{A}^n/n! \approx \sum_{n=0}^{N^*} \mathbf{A}^n/n!$ for some large number N^* (e.g., $N^* = 100$).

(c) Plot $P_{A,A}(t)$, $P_{A,B}(t)$ and $P_{A,C}(t)$ as functions of time t, for $t = 0, 1, 2, \ldots, 30$.
Remark: This will visualize the convergence to steady state for CTMC.

Reading: Read Green Ross Sections 6.5-6.6 and Exercise 6.16, 6.19, 6.24, 6.28 and 6.33 which have answers in the back. You can also try 6.14 and 6.23 if you need more exercises.

Appendix:
Numerical Solution of Stationary Distribution $\boldsymbol{\alpha}$

Consider a finite state space CTMC. When it is irreducible, we know that the DTMC is "good", since all states are positive-recurrent. Because time is now continuous, we no longer have to worry about the issue of (a)periodicity. Let the state space be $\mathcal{S} \equiv \{1, \ldots, n\}$. We know that the following *balance equation* (BE) has a unique solution.

$$\boldsymbol{\alpha}\,\mathbf{Q} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\alpha}\,\mathbf{e} = \sum_{i=1}^{n} \alpha_i = 1, \tag{1}$$

where the column vector $\mathbf{e} \equiv (1, \ldots, 1)^T \in \mathbb{R}^{n\times 1}$. When the transition rate matrix (TPM) $\mathbf{Q}$ is big, solving the BE by hand can be a headache! Let's now discuss on how to numerically solve the BE by programming (such as using MatLab).
First, note that there are n unknowns $(\alpha_1, \ldots, \alpha_n)$ but $(n+1)$ equations in (1). Inspired by HW4 which shows that one of the first n equations in $\boldsymbol{\pi}\,\mathbf{P} = \pi$ (BE for DTMC) is redundant, we claim the same holds here, namely, one of the first n equations in (1) here is again redundant (Try to prove this claim!). Let the row vector $\mathbf{e}_n \equiv (0, \ldots, 0, 1) \in \mathbb{R}^{1\times n}$. Without loss of generality, dropping the nth column of $\boldsymbol{\alpha}\,\mathbf{Q} = \mathbf{0}$ in (1) yields

$$\boldsymbol{\alpha}\;[\mathbf{Q}_1, \ldots, \mathbf{Q}_{n-1}|\mathbf{e}] = \mathbf{e}_n \in \mathbb{R}^{1\times n}, \tag{2}$$

where $\mathbf{Q}_k$ is the kth column of $\mathbf{Q}$, $k = 1, \ldots, n$. Hence, the steady state

$$\boldsymbol{\alpha} = \mathbf{e}_n\,\widehat{\mathbf{Q}}^{-1}, \quad \text{where} \quad \widehat{\mathbf{Q}} \equiv [\mathbf{Q}_1, \ldots, \mathbf{Q}_{n-1}|\mathbf{e}] \in \mathbb{R}^{n\times n}.$$

Hints for Assignment 11

6.12: Easy! This is a birth-and-death queue. Draw the rate diagram and mimic the treatment on the $M/M/1/3$ queue introduced in class.

6.13: (a) Easy! Mimic the treatment on the $M/M/1/3$ queue introduced in class.

(b) Easy! Mimic the treatment on the $M/M/1/3$ queue introduced in class.

(c) Easy! To measure the amount of business, compare the quantity computed in part (b).

OP 2: (a) Easy! No hint. At a second thought, just to be nice, it should be: *the long-run ....*

(b) Medium! First, note that $-Q_{i,i} = \gamma_i$, which is the rate of the exponential holding time at state i. Let $N_i(t)$ count the number of visits in state i by t and let $T_i(t) \equiv \int_0^t \mathbf{1}(X(s) = i)\, ds$ be the total length of time the CTMC is in state i during $[0, t]$. Consider

$$\frac{N_i(t)}{t} = \frac{T_i(t)}{t} \cdot \frac{N_i(t)}{T_i(t)}.$$

Then what will happen if we let $t \to \infty$? Mimic the argument introduced in class for DTMC. This kind of analysis should already become automatic.

(c) Easy! Not hint. At a second thought, just to be nice, it should be: *the long-run ....*

(d) Medium, but becomes easy provided part (a) and HW5 Problem 1. The idea is almost identical to HW5 Problem 1. Specifically, let $N_{i,j}(t)$ count the number of direct transitions from state i to state j by time t. Write $N_{i,j}(t) \equiv \sum_{k=1}^{N_i(t)} \mathbf{1}(***)$, where we still need help from our best-friend r.v.'s: indicators. We write

$$\frac{N_{i,j}(t)}{t} = \frac{\sum_{k=1}^{N_i(t)} \mathbf{1}(***)}{t} = \frac{N_i(t)}{t} \cdot \frac{\sum_{k=1}^{N_i(t)} \mathbf{1}(***)}{N_i(t)},$$

where the asymptotic limit (as $t \to \infty$) for the first term on the RHS is obtained already in (a) and (b). You should know how to proceed from here.

OP 3: (a) Easy! No hint.

(b) Medium! Let $\lfloor t \rfloor$ ($\lceil t \rceil$) be the biggest (smallest) integer that is $\leq t$ ($\geq t$). Start with the following "sandwish" argument:

$$\left(\frac{N(\lfloor t \rfloor)}{\lfloor t \rfloor}\right)\left(\frac{\lfloor t \rfloor}{\lceil t \rceil}\right) = \frac{N(\lfloor t \rfloor)}{\lceil t \rceil} \leq \frac{N(t)}{t} \leq \frac{N(\lceil t \rceil)}{\lfloor t \rfloor} = \left(\frac{N(\lceil t \rceil)}{\lceil t \rceil}\right)\left(\frac{\lceil t \rceil}{\lfloor t \rfloor}\right).$$

It is easy to see that the second term on the RHS goes to 1 as $t \to \infty$. We can next write the second term of the RHS as

$$\frac{1}{\lceil t \rceil} \sum_{k=1}^{\lceil t \rceil} Y_k,$$

where $Y_k \equiv N(k) - N(k-1)$. What can you observe for the sequence of r.v.'s $Y_1, Y_2, Y_3, \ldots$? Then what theorem do you have in mind? In the same way we can treat the LHS. If the LHS and RHS converge to the same limit as $t \to \infty$, according to a "sandwish" argument, the "meat" in the middle should converge to the same limit.

(c) Medium! First, split interval $[0, t]$ into two parts: $T_1(t)$ and $T_2(t)$, and write $N(t) \equiv N_1(t) + N_2(t)$, where $T_i(t) \equiv \int_0^t \mathbf{1}(X(s) = i)ds$ represents the length of time the CTMC is in state i during $[0, t]$, $N_i(t)$ counts the number of events that occurred when the system is in state i, $i = 1, 2$. Then we put all these pieces together

$$\frac{1}{t}N(t) = \left(\frac{T_1(t)}{t}\right) \cdot \left(\frac{N_1(t)}{T_1(t)}\right) + \left(\frac{T_2(t)}{t}\right) \cdot \left(\frac{N_2(t)}{T_2(t)}\right).$$

What happens if we let $t \to \infty$? Think about the interpretation of the stationary distribution for CTMC and use your result in (a).

(d) Medium! Following the hint for Part (c), we have $\mathbb{E}[N(t)|X(0) = 1] = \mathbb{E}[N_1(t)|X(0) = 1] + \mathbb{E}[N_2(t)|X(0) = 1]$. To compute $\mathbb{E}[N_i(t)|X(0) = 1]$, condition on $T_i(t)$, we get

$$\begin{aligned}
\mathbb{E}[N_i(t)|X(0) = 1] &= \mathbb{E}\left[\mathbb{E}[N_i(t)|T_i(t)]|X(0) = 1\right] \\
&= \lambda_i \, \mathbb{E}[T_i(t)|X(0) = 1] \\
&= \lambda_i \, \mathbb{E}\left[\int_0^t \mathbf{1}(X(s) = i)ds \,\middle|\, X(0) = 1\right] \\
&= \lambda_i \int_0^t \mathbb{P}(X(s) = i|X(0) = 1)\, ds = \lambda_i \int_0^t P_{1,i}(s)\, ds.
\end{aligned}$$

To obtain the probability in the integrand, read Example 6.11 in Green Ross.

OP 4: Medium! Common sense implies that the answer to (a) is obvious provided the question in (b). Part (c) is easy. Part (d) must be somewhat different with (b), otherwise there is no point.