Assignment 12: *Due on November 20*

CTMC: Time Reversibility and Markovian queueing networks, Renewal Counting Processes

From Green Ross: Do Exercise 6.26 in Chapter 6, and 7.1, 7.4, in Chapter 7.

Other Problems:

1. (**Jackson Queueing Networks**)
We now extend our analysis from the $M/M/1$ queue (which has only one queue, so too easy and not challenging) to the so-called *Jackson Queueing Network* (JQN):

  - Consider a system consisting of two $M/M/1$ queues in parallel.
  - Each queue has an *external* Poisson arrival process with rate $\lambda_i^{(0)}$, one server, and I.I.D. *exponential* service times with rate μ_i, $i = 1, 2$.
  - This network has an internal Markovian routing structure. For example, when a customer finishes service at server 1, with probability $P_{1,2} \geq 0$, he or she is routed to the end of the 2nd queue; with probability $P_{1,1}$, the customer goes back to the same queue, and with probability $P_{1,0} \equiv 1 - \sum_{j=1}^{2} P_{1,j} \geq 0$, he (she) leaves the system without coming back.
  - A JQN is denoted with notation $(M/M/1)^2/M$ in *queueing theory*, where the last $/M$ denotes the Markovian (i.e., probabilistic) routing policy. The model parameters for a JQN are the vector of external arrival rate $\boldsymbol{\lambda}^{(0)} \equiv (\lambda_1^{(0)}, \lambda_2^{(0)})$, the vector of service rate $\boldsymbol{\mu} \equiv (\mu_1, \mu_2)$ and the Markovian routing matrix $\mathbf{P}$. See Figure 1.

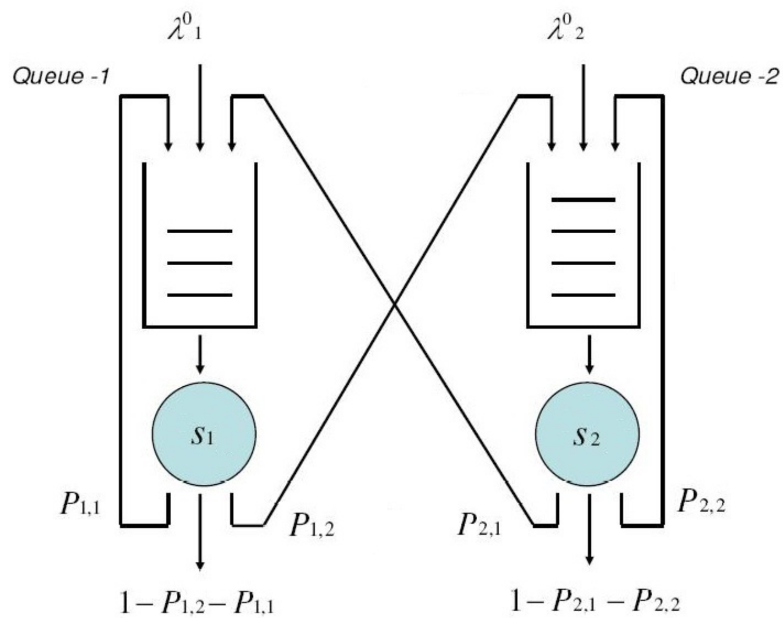

Figure 1: A Two-Queue Example of the Jackson Queueing Network (JQN).

Because of the assumptions on *exponential* service distribution and Poisson arrivals, it is obvious that we can model the system dynamics with a CTMC. We consider the two-dimensional *queue-length* process $\mathbf{Q}(t) \equiv (Q_1(t), Q_2(t))$, where $Q_i(t)$ is the number of customers in queue

i at time t, including the customer in service and those waiting in line (if any), $i = 1, 2$. There is no doubt that the two-dimensional process $\{\mathbf{Q}(t), t \geq 0\}$ is a CTMC.

Remark: This network queueing model was first identified and analyzed by the famous queueing theorist *James R. Jackson* (1924-2011). Here are some references:

[1] Jackson, J. Networks of waiting lines. *Operations Research* **4** 518–521. 1957.

[2] Jackson, J. Jobshop-like Queueing Systems. *Management Science* **10** 131–142. 1963.

Answer the following questions:

(a) What is the state space $\mathcal{S}$ of this CTMC? Is it a BAD process?

(b) Note that a state for this CTMC is an two-dimensional vector. Consider a state $\bar{\boldsymbol{n}} \equiv (n_1, n_2)$, meaning that $Q_i(t) = n_i$, $n_i = 0, 1, 2, \ldots$, $i = 1, 2$. Which states are accessible from state (n_1, n_2) in one transition? From which states is state (n_1, n_2) accessible in one transition?

(c) Construct the transition rate matrix (TRM) $\mathbf{Q}$ for this CTMC.

(d) We now want to determine the *stationary distribution* for this CTMC. In steady state, all queues are in equilibrium. Therefore, consider queue i, flow conservation implies that the *steady-state* total arrival rate λ_i at queue i (i.e., the sum of the external arrival rate $\lambda_i^{(0)}$ and the rate of feedbacks from other queues) must satisfy **one** of the following *traffic-rate equations* (TRE's):

$$\text{(i)} \quad \lambda_i = \lambda_i^{(0)} + \sum_{j=1}^{N} \lambda_j P_{ji}, \quad i = 1, 2;$$

$$\text{(ii)} \quad \lambda_i = \lambda_i^{(0)} + \sum_{j=1}^{N} \mu_j P_{ji}, \quad i = 1, 2.$$

So, which **one** is correct? Why?

(e) Recall that the *stationary distribution* vector $\bar{\boldsymbol{\alpha}} \equiv \big(\alpha_{(n_1,n_2)}, (n_1, n_2) \in \mathcal{S}\big)$ uniquely solves the balance equation $\bar{\boldsymbol{\alpha}}\, Q = 0$ and $\sum_{(n_1,n_2)\in\mathcal{S}} \alpha_{(n_1,n_2)} = 1$. Show that the *stationary distribution* is given by:

$$\alpha_{(n_1,n_2)} \equiv \mathbb{P}(Q_1(\infty) = n_1, Q_2(\infty) = n_2) = (1-\rho_1)\rho_1^{n_1} \times (1-\rho_2)\rho_2^{n_2}, \tag{1}$$

where $\rho_i \equiv \lambda_i/\mu_i$ is the *traffic intensity* for queue $i = 1, 2$, and λ_i satisfies the **correct** TRE in (d).

Remark: Equation (1) *has very intuitive implications:*

(1) The steady-state queue length $Q_i(\infty)$ for each queue i is Geometric with parameter $p_i = 1 - \rho_i$ (think about Problem 2 of IC-PS-12). Why? Think about how to get the marginal PMF from the joint PMF.

(2) The stationary distribution for JQN is the product of the stationary distribution for two $M/M/1$ queues.

(3) The steady-state queue-length for these 2 queues are 2 independent r.v.'s, i.e.

$$\mathbb{P}(Q_1(\infty) = n_1, Q_2(\infty) = n_2) = \mathbb{P}(Q_1(\infty) = n_1) \times \mathbb{P}(Q_2(\infty) = n_2),$$

which implies that the r.v.'s $Q_1(\infty)$ and $Q_2(\infty)$ are independent.

2. Two applications of JQN

   - (**The $\mathbf{Y}^2$ workshop**)
     Tasks arrive at a workshop in accordance with a Poisson process with rate 8. Each task will independently be routed to machine 1 operated by Yu-Ting (server 1) w.p. 5/8 and to machine 2 operated by Yi (server 2) w.p. 3/8. Each machine can only process one task at a time and the processing times of the two machines are independent exponential random variables with rate $\mu_1 = 25$ and $\mu_2 = 25$, respectively. However, 70% of the products processed by Yu-Ting and 90% of the products by Yi have flaws so that re-processing is needed. The defective products, once flow out of the machine, will equally likely be sent to either of the two machines.

     (a) After a long time, what is probability that Yu-Ting's machine has 2 tasks (one waiting and one being processed) and Yi's has 3 tasks?

     (b) What is the steady-state mean number of busy machines in the workshop?

     (c) What is the steady-state mean number of tasks in the workshop?

   - (**The $\mathbf{K}^2$ Barbershop revisited**)
     Treat Problem 2 of ICPS-12 as a special case of the JQN, with $\lambda_1^{(0)} = \lambda$, $\lambda_2^{(0)} = 0$, $P_{1,2} = 1$ and $P_{2,0} = 1$. Find the steady state distribution using part (e). Compare your result with the result introduced in class (based on time reversibility).
     Remark: the general structure of JQN can be used to represent all kinds of desired examples: tandem queues, parallel queues, queues with feedback, etc.

3. (**Joint CTMC with Independent Components**) If two CTMC's $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ are *independent* and *time reversible* with state spaces $\mathcal{S}^{\mathcal{X}}$ and $\mathcal{S}^{\mathcal{Y}}$, transition rate matrices Q^X and Q^Y, stationary distributions α^X and α^Y, respectively.

   (a) Find the transition rates for the two-dimensional processes $\{(X(t), Y(t)), t \geq 0\}$.

   (b) Find the stationary distribution for the two-dimensional processes $\{(X(t), Y(t)), t \geq 0\}$.

   (c) Is the two-dimensional process $\{(X(t), Y(t)), t \geq 0\}$ *time reversible* as well?

Reading: Read Green Ross Sections 7.1-7.4, 7.7 and 7.9. Examples 7.12, 7.13, 7.23, 7.24 and 7.25 are good examples which are very representative. If you need more exercises, you can also study Exercise 7.8, 7.22, 7.25 and 7.42 which have answers in the back.

Hints for Assignment 12

6.26: Make use of the *time reversibility* of this CTMC. This implies that the *reverse-time* CTMC has the same probability law as the original CTMC. Also note that "past departures" of the original CTMC are "future arrivals" of the *reverse-time* CTMC.

7.1: Note that it is possible to have multiple renewals occurring at the same time for a general RCP.

7.4: Note that to disprove something, one counterexample is enough. As an easy counterexample, how about an RCP with deterministic interarrival times?

OP 1: This is another big problem which is quite educational.

(a) Easy! No hint.

(b) Make use of the fact that the min of *Exp* r.v.'s is again *Exp* and events happens after independent *Exp* r.v.'s. There are the following *three* types of events:

  (i) an external arrival to queue i, $i = 1, 2$;

  (ii) a customer leaving the system from server i, $i = 1, 2$;

  (iii) a customer being routed inside the network from server i to queue j, $i \neq j$, $i, j = 1, 2$.

Let row vectors $e_1 \equiv (1, 0)$ and $e_2 \equiv (0, 1)$. Think about the meaning of states $\bar{\boldsymbol{n}} + e_i$, $\bar{\boldsymbol{n}} - e_i$, $\bar{\boldsymbol{n}} + e_i - e_j$ and $\bar{\boldsymbol{n}} - e_i + e_j$. However, you have to be careful: the cases $n_i = 0$ and $n_i \geq 1$ have to be treated differently.

(c) Just follow the analysis in (b).

(d) Just think about the question: Are we certain if a queue i is empty or it has least one customer in steady state? That would make a difference. Think about Problem 2 of IC-PS-12: what is the departure rate of the first queue there? Now for the JQN, what are the steady-state departure rate for each queue i?

(e) Medium (well)! Theoretically speaking, you just have to check the balance equation: $\bar{\boldsymbol{\alpha}}\, Q = 0$ because it should have a unique solution. So this seems to be a "plug-in" problem. However, you will definitely hate me if this is what I ask you to do! Note that Q is a HUGE matrix of infinite dimension! Fortunately, the rate matrix Q is sparse. We also know that an alternative representation of the balance equation is "*rate in = rate out*" for each state $\bar{\boldsymbol{n}} \in \mathcal{S}$. Therefore, it remains to specify: (i) from which states can the CTMC go to state $\bar{n}$ in one step and (ii) to which states can the CTMC go from state $\bar{n}$ in one step. But we already know the answer from part (b) and the rates of these transitions from (c). It remains the verify the equality:

$$\text{“long-run rate into state } \bar{\boldsymbol{n}}\text{”} = \sum_{\bar{\boldsymbol{n}}'} \alpha_{\bar{\boldsymbol{n}}'}\, Q_{\bar{\boldsymbol{n}}', \bar{\boldsymbol{n}}} = \sum_{\bar{\boldsymbol{n}}''} \alpha_{\bar{\boldsymbol{n}}}\, Q_{\bar{\boldsymbol{n}}, \bar{\boldsymbol{n}}''} = \text{“long-run rate leaving } \bar{\boldsymbol{n}}\text{”}.$$

Write out the above equation by plugging in different terms and check if LHS = RHS. Also, make use of the correct TRE in (d). What remains from here is just algebra.

OP 3: (a) Let $i_1, i_2 \in \mathcal{S}^{\mathcal{X}}$ and $j_1, j_2 \in \mathcal{S}^{\mathcal{Y}}$. Can the 2-D process jump from state (i_1, j_1) to state (i_2, j_2) in one step? If not, what are the intermediate states?

(b) Make use of the independence.

(c) Check the definition (or the criterion equation) of *time reversibility.*