1. Pooh Bear and the Three Honey Trees
A bear of little brain named Pooh is fond of honey. Bees producing honey are located in three trees: tree A, tree B and tree C. Tending to be somewhat forgetful, Pooh goes back and forth among these three honey trees randomly (in a Markovian manner) as follows:

- from A, Pooh goes next to B or C with probability 1/2;
- from B, Pooh goes next to A w.p. 3/4 and to C w.p. 1/4;
- from C, Pooh always goes next to A.

(a) Construct a DTMC and specify the transition probabilities.

(b) Find the limiting proportion of time that Pooh spends at each honey tree.

(c) Is this DTMC time-reversible?
2. The weather problem revisited (transforming a non-DTMC into a DTMC) Now assume that the weather on one day depends on weathers of the previous two days.

- if it has rained for the past two days, then it will rain tomorrow w.p. 0.7;
- if it rained today but not yesterday, then it will rain tomorrow w.p. 0.5;
- if it rained yesterday but not today, then it will rain tomorrow w.p. 0.4;
- if it has not rained in the past two days, then it will rain tomorrow w.p. 0.2.

Let $X_n$ denote the weather condition on day $n$. Is $\{X_n, n \geq 0\}$ a DTMC? If not, how do we build a DTMC?
3. **The Markov Mouse**

We started by considering how to model a mouse moving around in a maze. The maze is a closed space containing nine rooms. The space is arranged in a three-by-three array of rooms, with doorways connecting the rooms, as shown in the figure below. There are doors leading to adjacent rooms. We assume that the mouse is a Markov mouse, i.e., the mouse moves randomly from room to room, with the probability distribution of the next room depending only on the current room, not on the history of how it got to the current room. (This is the Markov property.) Moreover, we assume that the mouse is equally likely to choose each of the available doors in the room it occupies.

We now model the movement of the mouse as a Markov chain. The state of the Markov chain is the room occupied by the mouse. We let the time index \( n \) refer to the \( n \)th room visited by the mouse. So we make a discrete-time Markov chain. We let \( X_n \) be the state (room) occupied by the mouse on step (or time or transition) \( n \). The initial room is \( X_0 \). The room after the first transition is \( X_1 \), and so forth. Then \( \{X_n : n \geq 0\} \) is a DTMC, i.e., a discrete-time discrete-state Markov process.

(a) Find the transition probability matrix \( P \).
(b) Find the following probabilities:

\[ P_{1,1}^{(3)} = \mathbb{P}(X_3 = 1|X_0 = 1) =? \]
\[ P_{1,1}^{(7)} = ? \]
\[ P_{1,9}^{(7)} = ? \]
\[ P_{2,4}^{(7)} = ? \]
\[ P_{2,8}^{(17)} = ? \]
\[ P_{1,1}^{(17)} = ? \]

(c) Find the following probabilities:

\[ P_{2,2}^{(18)} = ? \]
\[ P_{1,2}^{(17)} = ? \]
\[ P_{1,2}^{(19)} = ? \]
\[ P_{4,2}^{(18)} = ? \]
\[ P_{2,2}^{(18)} = ? \]
\[ P_{1,1}^{(18)} = ? \]
\[ P_{2,1}^{(28)} = ? \]
\[ P_{1,7}^{(28)} = ? \]
\[ P_{3,9}^{(28)} = ? \]
4. **Gambler’s Ruin Problem**

Consider a gambler who at each play of the game has probability \( p \) of winning 1 unit and probability \( q = 1 - p \) of losing 1 unit. Assuming successive plays of the game are independent, what is the probability that starting with \( i \) units, the gambler’s fortune will reach \( n \) before reaching 0.