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An Online Learning Approach to Dynamic Pricing and Capacity Sizing in Service Systems

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We study a dynamic pricing and capacity sizing problem in a $GI/GI/1$ queue, where the service provider’s objective is to obtain the optimal service fee $p$ and service capacity $\mu$ so as to maximize cumulative expected profit (the service revenue minus the staffing cost and delay penalty). Due to the complex nature of the queueing dynamics, such a problem has no analytic solution so that previous research often resorts to heavy-traffic analysis in that both the arrival rate and service rate are sent to infinity. In this work we propose an online learning framework designed for solving this problem which does not require the system’s scale to increase. Our algorithm organizes the time horizon into successive operational cycles and prescribes an efficient procedure to obtain improved pricing and staffing policies in each cycle using data collected in previous cycles. Data here include the number of customer arrivals, waiting times, and the server’s busy times. The ingenuity of this approach lies in its online nature, which allows the service provider do better by interacting with the environment. Effectiveness of our online learning algorithm is substantiated by (i) theoretical results including the algorithm convergence and regret analysis (with a logarithmic regret bound), and (ii) engineering confirmation via simulation experiments of a variety of representative $GI/GI/1$ queues.

Key words: online learning in queues; service systems; capacity planning; staffing; pricing in service systems

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1. Introduction

In this paper, we study a service queueing model where the service provider manages congestion and revenue by dynamically adjusting the price and service capacity. Specifically,
we consider a $GI/GI/1$ queue, in which the demand for service is $\lambda(p)$ per unit of time when each customer is charged by a service fee $p$; the cost for providing service capacity $\mu$ is $c(\mu)$; and a holding cost $h_0$ incurs per job per unit of time. By choosing the appropriate service fee $p$ and capacity $\mu$, the service provider aims to maximize the net profit, which is the service fee minus the staffing cost and penalty of congestion, i.e.,

$$\max_{\mu, p} \mathcal{P}(\mu, p) \equiv p\lambda(p) - c(\mu) - h_0\mathbb{E}[Q_\infty(\mu, p)],$$

where $Q_\infty(\mu, p)$ is the steady-state queue length under service rate $\mu$ and price $p$.

Such a problem has a long history, see for example Kumar and Randhawa (2010), Lee and Ward (2014), Lee and Ward (2019), Maglaras and Zeevi (2003), Nair et al. (2016), Kim and Randhawa (2018) and the references therein. Due to the complex nature of the queueing dynamics, exact analysis is challenging and often unavailable (computation of the optimal dynamic pricing and staffing rules is not straightforward even for the Markovian $M/M/1$ queue Ata and Shneorson (2006)). Therefore, researchers resort to heavy-traffic analysis to approximately obtain performance evaluation and optimization results. Commonly adopted heavy-traffic regimes require sending the arrival rate and service capacity (service rate or number of servers) to $\infty$. The main advantage of the asymptotic analysis is its tractability, because convenient approximation formulas are often available in the heavy-traffic limit (one no longer needs to deal with the discrete nature of the queueing performance functions). Although heavy-traffic analysis provides satisfactory results for large-scale queueing systems (e.g., models for customer contact centers), approximation formulas based on heavy-traffic limits usually become inaccurate as the system scale decreases. Another potential restriction of solution techniques based on heavy-traffic theories is that they often have to assume the full (or partial) knowledge of the distributions of the service and interarrival times. For example, solutions based on the functional central limit theorem (FCLT) limits require at least the knowledge of the first two moments of relevant random variables (Whitt 2006, Liu and Whitt 2014).

In this paper we propose an online learning framework designed for solving Problem (1). In contrast to the aforementioned heavy-traffic method, our online learning approach has two main appeals: (i) It does not require any asymptotic scaling so it can apply to service systems having smaller scales; and (ii) it does not require the distributional information of the service and interarrival times beyond their means, hence this is a more robust
method. According to our online learning algorithm, the $GI/GI/1$ queue will be operated in successive cycles, where in each cycle the service provider’s decisions on the service fee $p$ and service capacity $\mu$, deemed the best by far, are obtained using the system’s data collected in previous operational cycles. Data hereby include (i) the number of customers who join for service, (ii) customers’ waiting times, and (iii) the server’s busy time, which are all easy to collect. Newly generated data, which represent the response from the (random and complex) environment to the present operational decisions, will be used to obtain improved pricing and staffing policies in the next cycle. In this way the service provider can dynamically interact with the environment so that the operational decisions can evolve and eventually approach the optimal solutions to Problem (1).

At the beginning of each cycle $k$, the service provider’s decisions $(p_k, \mu_k)$ will be computed and enforced throughout the cycle. At the heart of our procedure for computing $(p_k, \mu_k)$ is to obtain an improved estimator for $H_{k-1}$, the gradient of the objective function of (1), using past experience. Specifically, our online algorithm will update $(p_k, \mu_k)$ according to

$$ (\mu_k, p_k) \leftarrow (\mu_{k-1}, p_{k-1}) + \eta_{k-1} H_{k-1}, $$

where $\eta_k$ is the updating step size for cycle $k$. Besides showing that, under our online learning scheme, the decisions in cycle $k$, $(p_k, \mu_k)$ will converge to the optimal solutions $(p^*, \mu^*)$ as $k$ increases, we quantify the effectiveness of our method by computing the regret - the cumulative loss of profit due to the suboptimality of $(p_k, \mu_k)$, namely, the maximum profit under the optimal strategy minus the expected profit earned according to the online algorithm by time $t$. When the algorithm parameters are chosen optimally, we show that our regret bound is logarithmic so that the service provider, with any initial pricing and staffing policy $(p_0, \mu_0)$, will quickly learn the optimal solutions without losing much profit in the learning process.

Main challenges of online learning in queueing systems. A crucial step of our online learning method is to develop effective ways to control the nonstationary error that arises at the beginning of every cycle due to the policy update. Towards this, we develop a new regret analysis framework for the transient queueing performance that not only helps establish desired regret bounds for the specific online $GI/GI/1$ algorithm, but may also be used to develop online learning method for other queueing models (see Section 3). Another challenge we have to address here is to devise a convenient gradient estimator for the online
learning algorithm (specifically, an estimator for the gradient of \( \mathbb{E}[Q_{\infty}(\mu, p)] \)). The estimator should have a negligible bias to warrant a quick convergence of the algorithm, and at the same time, its computation (using previous data) should be sufficiently straightforward to ensure the ease of implementation (The detailed gradient estimator for the GI/GI/1 system is given in Section 4).

1.1. Related Literature

The present paper is related to the following three streams of literature.

**Pricing and capacity sizing in queues.** There is a rich literature on pricing and capacity sizing in service systems under different settings. Maglaras and Zeevi (2003) studies pricing and capacity sizing problem in a processor sharing queue motivated by internet applications; Kumar and Randhawa (2010) considers a single-server system with nonlinear delay costs; Nair et al. (2016) studies M/M/1 and M/M/k systems with network effect among customers; Kim and Randhawa (2018) considers a dynamic pricing problem in a single-server system. The specific problem (1) we consider here is most closely related to Lee and Ward (2014), i.e. joint pricing and capacity sizing for the GI/GI/1 queue. Later, the authors extend their results to the GI/GI/1 + G model with customer abandonment in Lee and Ward (2019). As there is usually no closed-form solution for the optimal strategy or equilibrium, asymptotic analysis is adopted under large-market assumptions. In detail, both the capacity and the potential demand are scaled to infinity, and then, some limiting property, for instance, the ratio between endogenous demand and capacity, is studied under the scaling, in order to obtain economical and managerial insights. Most of the papers conclude that heavy-traffic regime is economically optimal. (There are some exceptions where heavy-traffic regime is not optimal, for example, Kumar and Randhawa (2010) shows that agent is forced to decrease its utilization if the delay cost is concave.) Our algorithm is motivated by the pricing and capacity sizing problem for service systems, however, as explained previously, our methodology is very different from the asymptotic analysis used in these papers.

**Reinforcement learning for queueing systems.** Our paper is also related to a small but growing literature on reinforcement learning (RL) for queueing systems. Dai and Gluzman (2020) studies an actor-critic algorithm for queueing networks. Liu et al. (2019) and Shah et al. (2020) develop RL techniques to treat the unboundedness of the state space of
queueing systems. In some sense, the algorithm developed in the present paper may be viewed as a simplified version of the policy gradient method, a special class a RL methods, see Remark 2 for detailed discussions. Distinct from the above-mentioned literature which focuses on methodological development for general models, our algorithm design and its regret analysis take advantage of the specific queueing system structure so as to establish tractable regret bounds and more accurate control of the convergence rate.

Stochastic gradient decent algorithms. In general, our algorithm falls into the broad class of stochastic gradient descent (SGD) algorithms. There are some early papers on SGD algorithms for steady-state performance of queues (see Fu (1990), Chong and Ramadhe (1993), L’Ecuyer et al. (1994), L’Ecuyer and Glynn (1994) and the references therein). In particular, these papers have established convergence results of SGD algorithms for capacity sizing problems with a variety of gradient estimating designs. In this paper, we consider a more general setting in which the price is also optimized jointly with the service capacity. Besides, in order to establish theoretical bounds for the regret, we conduct a careful analysis on the convergence rate of the algorithm and provide an explicit guidance for the optimal choice of algorithm parameters, which is not discussed in this early literature. Our algorithm design and analysis are also related to the online learning methods in recent inventory management literature (Burnetas and Smith 2000, Huh et al. 2009, Huh and Rusmevichientong 2013, Zhang et al. forthcoming, Yuan et al. 2020). Among these papers, our work is perhaps most closely related to Huh et al. (2009) where the authors develop an SGD based learning method for an inventory model with bounded replenishment lead time. To control the bias and variance of the gradient estimator, they characterize the system’s nonstationary performance. Still, due to the unique natures of queueing models, we develop a new regret analysis framework as we shall explain with details in Section 1.2.

1.2. Contributions and Organizations:
We summarize our contributions below.

- To the best of our knowledge, the present work is the first to develop an online learning framework (for pricing and staffing) in a queueing system. Due to the complex nature of queueing systems, previous research often resorts to asymptotic heavy-traffic analysis to approximately solve for desired operational decisions. The ingenuity of our online learning method lies in the ability to obtain the optimal solutions without needing the
system scale (e.g., arrival rate and service rate) to grow large. The other appeal of our method is its robustness, especially in its independence on the detailed distributions of service and arrival times.

- A critical step of the regret analysis is the treatment of the transient system dynamics, because when improved operational decisions are obtained and implemented at the beginning of a new period, the queueing performance will shift away from previously established steady-state performance. Towards this, we develop a new way to treat and bound the transient queueing performance in the regret analysis of our online learning algorithm. Comparing to previous literature (e.g., the regret bound is $O(T^{2/3})$ in Huh et al. (2009)), our analysis of the regret due to nonstationarity gives a much tighter logarithmic bound. We believe that the regret analysis in the present paper may be extended to other queueing systems which share similar properties to $GI/GI/1$.

- Supplementing the theoretical results of our regret bound, we evaluate the practical effectiveness of our method by conducting comprehensive simulation experiments; we show that our online algorithm is effective for several representative $GI/GI/1$ queues, even when certain model assumptions are relaxed. In addition, we compare optimal solutions generated from the online learning algorithm to previously established results based on heavy-traffic limits of the queueing model.

**Organization of the paper.** In Section 2, we introduce the model assumptions and provide an outline of our online learning algorithm. In Section 3, we conduct the regret analysis for the online algorithm by separately treating the *regret of nonstationarity* - the part of regret arising from the transient system dynamics, and the *regret of suboptimality* - the part originating from the errors due to suboptimal pricing and staffing decisions. In Section 4, we give the detailed description of the online algorithm; we also establish a logarithmic regret bound by appropriately selecting our algorithm parameters. In Section 5 we conduct numerical experiments to confirm the effectiveness of the algorithm. We draw concluding remarks in Section 6. In the e-companion, we give all technical proofs and provide additional numerical examples.

## 2. Problem Setting and Algorithm Outline

In Section 2.1 we first describe the queueing model and technical assumptions. In Section 2.2, we provide a general outline of our online SGD algorithms. Finally, in Section 2.3 we conduct preliminary analysis of the queueing performance under the online SGD algorithm.
2.1. Model and Assumptions

We study a GI/GI/1 queueing system having customer arrivals according to a renewal process with generally distributed interarrival times (the first GI), independent and identically distributed (i.i.d.) service times following a general distribution (the second GI), and a single server that provides service in the first-in-first-out (FIFO) discipline. Each customer upon joining the queue is charged by the service provider a fee $p > 0$. The demand arrival rate (per time unit) depends on the service fee $p$ and is denoted as $\lambda(p)$. To maintain a service rate $\mu$, the service provider continuously pays a staffing cost at a rate $c(\mu)$ per time unit.

With $\mu \in [\underline{\mu}, \bar{\mu}]$ and $p \in [\underline{p}, \bar{p}]$, the service provider’s goal is to determine the optimal service fee $p^*$ and service capacity $\mu^*$ with the objective of maximizing the steady-state expected profit (1), or equivalently minimizing the objective function $f(\mu, p)$ as follows

$$
\min_{(\mu, p) \in B} f(\mu, p) \equiv h_0 \mathbb{E}[Q_\infty(\mu, p)] + c(\mu) - p\lambda(p), \quad B \equiv [\underline{\mu}, \bar{\mu}] \times [\underline{p}, \bar{p}].
$$

We shall impose the following assumptions on the above service system throughout the paper.

**Assumption 1.** *(Demand rate, staffing cost, and uniform stability)*

(a) The arrival rate $\lambda(p)$ is continuously differentiable and non-increasing in $p$.  
(b) The staffing cost $c(\mu)$ is continuously differentiable and non-decreasing in $\mu$.  
(c) The lower bounds $\underline{p}$ and $\underline{\mu}$ satisfy that $\lambda(p) < \underline{\mu}$ so that the system is uniformly stable for all feasible choices of the pair $(\mu, p)$.

Part (c) of Assumption 1 is commonly used in the literature of SGD methods for queueing models to ensure that the steady-state mean waiting time $\mathbb{E}[W_\infty(\mu, p)]$ is differentiable with respect to decision parameters (see Chong and Ramadge (1993), Fu (1990), L’Ecuyer et al. (1994), L’Ecuyer and Glynn (1994), also see Theorem 3.2 of Glasserman (1992)). In the our numerical experiments (see Section EC.2.1), we show that our online algorithm remains effective when this assumption is relaxed.

In this study, we emphasize that our algorithm depends on the arrival process and service times only through their first moment characteristics $\lambda(p)$ and $\mu$, that is, we do not require the detailed knowledge of their full distributions. But in order to develop explicit bounds for the part of the regret due to the nonstationarity of the queueing processes,
we require both distributions to be light-tailed. Specifically, since the actual service and interarrival times are subject to our pricing and staffing decisions, we model the interarrival and service times by two scaled random sequences \( \{U_n/\lambda(p)\} \) and \( \{V_n/\mu\} \), where \( U_1, U_2, \ldots \) and \( V_1, V_2, \ldots \) are two independent i.i.d. sequences of random variables having unit means, i.e., \( \mathbb{E}[U_n] = \mathbb{E}[V_n] = 1 \). We make the following assumptions on \( U_n \) and \( V_n \).

**Assumption 2. (Light-tailed Service and interarrival)**

There exists a constant \( \eta > 0 \) such that the moment-generating functions

\[
\mathbb{E}[\exp(\eta V_n)] < \infty \quad \text{and} \quad \mathbb{E}[\exp(\eta U_n)] < \infty.
\]

In addition, there exists \( 0 < \theta < \eta/2, 0 < a < (\mu - \lambda(p))/(\mu + \lambda(p)) \) and \( \gamma_0 > 0 \) such that

\[
\phi_U(-\theta) < -(1-a)\theta - \gamma_0 \quad \text{and} \quad \phi_V(\theta) < (1+a)\theta - \gamma_0,
\]

(3)

where \( \phi_V(\theta) \equiv \log \mathbb{E}[\exp(\eta V_n)] \) and \( \phi_U(\theta) \equiv \log \mathbb{E}[\exp(\eta U_n)] \) are the cumulant generating functions of \( U \) and \( V \).

Note that \( \phi'_U(0) = \phi'_V(0) = 1 \) as \( \mathbb{E}[U] = \mathbb{E}[V] = 1 \). So (3) holds as long as the functions \( \phi_U \) and \( \phi_V \) are smooth around 0, which is true for many distributions of \( U \) and \( V \) used in common queueing models. Assumption 2 will be used in our proofs to build explicit bound for the regret of nonstationarity.

Finally, in order to warrant the convergence of our online learning algorithm, we require a convex structure for the problem in (2), which is common in the SGD literature; see Broadie et al. (2011), Kushner and Yin (2003) and the references therein.

Let \( x^* \equiv (\mu^*, p^*) \) and \( x \equiv (\mu, p) \). Let \( \nabla f(x) \) denote the gradient of a function \( f(x) \) and \( \| \cdot \| \) denote the Euclidean norm.

**Assumption 3. (Convexity and Smoothness)**

There exist finite positive constants \( K_0 \leq 1 \) and \( K_1 > K_0 \) such that for all \( x \in \mathcal{B} \)

\[
(a) \quad (x - x^*)^T \nabla f(x) \geq K_0 \| x - x^* \|^2;
\]

(b) \( \| \nabla f(x) \| \leq K_1 \| x - x^* \| \);

**Remark 1.** Since \( \mathcal{B} \) is a compact set, condition (b) holds as long as the objective function \( f \) is continuously differentiable in the second order. Although the verification of Assumption 3, especially Condition (a), may not be straightforward for a general GI/GI/1 model, it is
possible to obtain some convenient sufficient conditions. For example, under Assumption 1, one sufficient condition for (a) is to require that \( c(\mu) \) be convex and \( \lambda'(p) + p\lambda(p) \leq 0 \) for \( p \in [\underline{p}, \bar{p}] \). We refer the readers to Section EC.3 for detailed discussions of Assumption 3. In addition, we conduct simulations in Sections 5 and EC.2 to show that our algorithm works effectively for \( GI/GI/1 \) queues with representative service and interarrival distributions even though verification of Assumption 3 is not straightforward.

2.2. Outline of the Online SGD Algorithm

In general, an SGD algorithm for a minimization problem \( \min_x f(x) \) over a compact set \( B \) relies on updating the decision variable via the recursion

\[
x_{k+1} = \Pi_B (x_k - \eta_k H_k), \quad k \geq 1.
\]

where \( \{\eta_k\} \) is a given sequence of constants, \( H_k \) is a random estimator for \( \nabla f(x_k) \), \( x_k \) is the optimal decision by step \( k \), and the operator \( \Pi_B \) restricts the updated decision in \( B \). For problem (2), we let \( x_k \equiv (\mu_k, p_k) \) represent the service capacity and price at step \( k \). We define

\[
B_k \equiv \max_{x \in B} \|\mathbb{E}[H_k - \nabla f(x_k)|x_k = x]\| \quad \text{and} \quad \mathcal{V}_k \equiv \max_{x \in B} \mathbb{E}[\|H_k\|^2|x_k = x],
\]

as the upper bounds for the bias and second moment of the gradient estimator \( H_k \). As we shall see later, \( B_k \) and \( \mathcal{V}_k \) play an important role in optimally designing the algorithm and establishing desired regret bounds.

The standard SGD algorithm iterates in discrete step \( k \). In our setting, however, the queueing system and objective function \( f(\mu, p) \) are defined in continuous time (in particular, \( Q_\infty(\mu, p) \) is the steady-state queue length observed in continuous time). To facilitate the regret analysis, we first transform the objective function into an expression of customer waiting times that are observed in discrete time. By Little’s law, we can rewrite the objective function \( f(\mu, p) \) as, for all \( (\mu, p) \in B \),

\[
f(\mu, p) = h_0 \lambda(p) \left( \mathbb{E}[W_\infty(\mu, p)] + \frac{1}{\mu} \right) + c(\mu) - p\lambda(p),
\]

where \( W_\infty(\mu, p) \) is the steady-state waiting time under \( (\mu, p) \). In each cycle \( k \), our algorithm adopts the average of \( D_k \) observed customer waiting times to estimate \( \mathbb{E}[W_\infty(\mu, p)] \), where \( D_k \) denotes the number of customers that enter service in cycle \( k \) (we call \( D_k \) the cycle
length of cycle $k$). But any finite $D_k$ will introduce a non-negligible bias to our gradient estimate $H_k$. To mitigate the bias due to the transient performance of the queueing process, we shall let the cycle length $D_k$ be increasing in $k$ (in this way the transient bias can vanish eventually). We give the outline of the algorithm below.

**Outline of Online SGD Algorithm:**

0. Input: $\{D_k\}$ and $\{\eta_k\}$ for $k = 1, 2, \ldots, L$, initial policy $x_1 = (\mu_1, p_1)$.

For $k = 1, 2, \ldots, L$,

1. Operate the $GI/GI/1$ queue under policy $x_k = (\mu_k, p_k)$ until $D_k$ customers enter service in the $k^{th}$ cycle.

2. Collect and use the data (e.g., customer delays) to build an estimator $H_k$ for $\nabla f(\mu_k, p_k)$.

3. Update $x_{k+1} = \Pi_B(x_k - \eta_k H_k)$.

**Remark 2 (Online learning vs. reinforcement learning).** The online nature of this algorithm makes it possible to obtain improved decisions by learning from past experience, which is in the spirit of the essential ideas of reinforcement learning where an agent (hereby the service provider) aims to tradeoff between exploration (Step 1) and exploitation (Steps 2 and 3). Effectiveness of the algorithms lies in properly choosing the algorithm parameters and devising an efficient gradient estimator $H_k$. For example, if $D_k$ is too small we may not be able to generate sufficient data (we do not exploit enough); if $D_k$ is too large we may lose too much profit due to suboptimality of the present policy (we do not explore enough). In particular, this online-SGD algorithm may be viewed as a special case of the policy gradient (PG) algorithm - a subclass of RL methods which aims to fit the RL policy by convenient parametric models (the general idea of PG is to estimate the policy parameters using the gradient of the value function learned via continuous interaction with the system, see for example Sutton and Barto (2018)). To put this into perspective, the policy in the present paper is specified by a pair of parameters $(\mu, p)$, and in each iteration, we update the policy parameters using an estimated policy gradient $H_k$ learned from data of the queueing model. In the subsequent sections, we give detailed regret analysis that can be used to establish optimal algorithm parameters (Section 3) and develop an efficient gradient estimator (Section 4).
2.3. System Dynamics under the Online Learning Algorithm

We explain explicitly the dynamic of the queueing system under our online optimization algorithm. We first define notations for relevant performance functions. For \( k \geq 1 \), let \( \lambda_k \equiv \lambda(p_k) \) and \( T_k \) be length of cycle \( k \) (in the units of time rather than number of customers). For \( n = 0, 1, 2, ..., D_k \), let \( W^k_n \) be the waiting time of the \( n^{th} \) customer that begins service in cycle \( k \), with \( W^k_0 \equiv W^{k-1}_{D_{k-1}} \). Define \( Q^k_n \) as the length of the waiting queue left by the \( n^{th} \) customer upon entering service in cycle \( k \). Recall that \( D_k \) is the number of customers that enter service in cycle \( k \), define \( Q_k \) as the number of remaining customers in the system right after the \( D_{k-1}^{th} \) customer in cycle \( k - 1 \) enters service. We use the two i.i.d. random sequences \( V^k_n \) and \( U^k_n \) to construct the service and inter-arrival times in cycle \( k \), \( n = 1, 2, ..., D_k \).

The detailed dynamics of the queueing system in cycle \( k \) is summarized as follows:

- **Control policy.** In cycle \( k \), we adopt the pricing and staffing decisions \((p_k, \mu_k)\). Cycle \( k \) ends once a total number of \( D_k \) (of which the value is to be determined later) customers complete services, and a new cycle \( k + 1 \) begins with new policy \((p_{k+1}, \mu_{k+1})\).

- **Service times.** The service time of customer \( n \) is \( S^k_n = V^k_{n+1}/\mu_k \). In particular, the last customer in cycle \( k - 1 \) receives service in cycle \( k \), so \( S^{k-1}_{D_{k-1}} \equiv S^k_0 = V^k_1/\mu_k \).

- **Leftovers from previous cycle.** There are \( Q_k \) existing customers at the beginning of cycle \( k \), with \( Q_k - 1 \) customers waiting in queue and customer 0 in service. They arrived in the previous cycle with rate \( \lambda_{k-1} \).

- **New arrivals.** The inter-arrival time \( \tau^k_n \) between customer \( n - 1 \) and \( n \) in cycle \( k \) is \( \tau^k_n = U^k_n/\lambda_{k-1} \) for \( n = 1, 2, ..., Q_k \), and \( U^k_n/\lambda_k \) for \( n = Q_k + 1, ..., D_k \).

- **Service fee.** The initial \( Q_k \) customers pay a fee \( p_{k-1} \) (because they arrived in the previous cycle) while the all new arrivals pay \( p_k \).

- **Customer delay.** Customers’ waiting times in cycle \( k \) are characterized by the recursions

\[
W^k_n = \begin{cases} 
(W^k_{n-1} + \frac{V^k_{n}}{\mu_k} - \frac{U^k_{n}}{\lambda_{k-1}})^+ & \text{for } 1 \leq n \leq Q_k; \\
(W^k_{n-1} + \frac{V^k_{n}}{\mu_k} - \frac{U^k_{n}}{\lambda_k})^+ & \text{for } Q_k + 1 \leq n \leq D_k.
\end{cases}
\]

where \( x^+ \equiv \max\{x, 0\} \).
• **Server’s busy time.** The age of the server’s busy time observed by customer $n$ upon arrival, which is the length of time the server has been busy since the last idleness, is given by the recursions

$$X_n^k = \begin{cases} 
X_{n-1}^k + \frac{U_n^k}{\lambda_n^k} \mathbf{1}_{\{W_n^k > 0\}} & \text{for } 1 \leq n \leq Q_k; \\
X_{n-1}^k + \frac{U_n^k}{\lambda_n^k} \mathbf{1}_{\{W_n^k > 0\}} & \text{for } Q_k + 1 \leq n \leq D_k.
\end{cases}$$

where $X_0^k = X_{D_k-1}^k$, \hspace{1cm}(7)

Remark 3. We provide explanations for (6) and (7). First, recursion (6) simply follows from Lindley’s equation. Next, recursion (7) follows from the observation that, for customer $n$, if the queue is empty upon its arrival, the observed busy time is simply 0 by definition; otherwise, the server must have been busy since the arrival of the previous customer and therefore, the observed busy time by customer $n$ should extend that of customer $n - 1$ by an additional inter-arrival time. As we shall see later, both the delay and busy time observed by customers will be important ingredients (i.e., data) for building the gradient estimator of the online learning algorithm.

We end this section by providing a uniform boundedness result for all relevant queueing functions. This result below will be used in the next sections to establish desired regret bounds. The proof follows a stochastic ordering approach and is given in Section EC.1.1.

Lemma 1. *(Uniform boundedness of relevant queueing functions)*

Under Assumptions 1 and 2, there exists a finite positive constant $M > 0$ such that for any sequences $(\mu_k, p_k) \in \mathcal{B}$ and $D_k \geq 1$, we have, for all $k \geq 1$, $1 \leq n \leq D_k$ and $1 \leq m \leq 4$, and $\eta > 0$ as defined in Assumption 2,

$\mathbb{E}[(W_n^k)^m], \mathbb{E}[(X_n^k)^m], \mathbb{E}[(Q_n^k)^m], \mathbb{E}[\exp(\eta W_n^k)]$ and $\mathbb{E}[\exp(\eta Q_n^k)]$

are all bounded by $M$.

3. **Regret Analysis**

In this section, we conduct the regret analysis for the online SGD algorithms described in Section 2.2. First, we write the overall regret as the sum of two terms: regret of nonstationarity which quantifies the error due to the system’s transient performance, and regret of suboptimality which accounts for the suboptimality error due to present policies.
According to the system dynamics, the expected net cost of the queueing system incurred in cycle $k$ is

$$\rho_k = \mathbb{E} \left[ \sum_{n=1}^{Q_k} (h_0(W_n^k + S_n^k) - p_{k-1}) + \sum_{n=Q_k+1}^{D_k} (h_0(W_n^k + S_n^k) - p_k) + c(\mu_k)T_k \right].$$

The regret $R_k$ in cycle $k$ is the expected system cost in cycle $k$ minus the optimal cost, specifically,

$$R_k = \rho_k - f(\mu^*, p^*) = \mathbb{E}[T_k] - f(\mu_k, p_k).$$

Intuitively, $R_{1,k}$ measures performance error due to transient queueing dynamics (regret of nonstationarity), while $R_{2,k}$ accounts for the suboptimality error of control parameters $(\mu_k, p_k)$ (regret of suboptimality). Consequently, the total regret accumulated in the first $L$ cycles is

$$R(L) = \sum_{k=1}^{L} R_k = \sum_{k=1}^{L} R_{1,k} + \sum_{k=1}^{L} R_{2,k} \equiv R_1(L) + R_2(L).$$

In what follows, we characterize the two terms $R_1(L)$ and $R_2(L)$ separately. To treat $R_1(L)$, we develop in Section 3.1 a new framework to analyze the transient behavior of the queueing systems using techniques of coupling (Theorem 1). The development of the theoretical bound for $R_2(L)$ is given in Section 3.2 (Theorem 2). Results in these sections provide convenient conditions that facilitate the convergence analysis and regret bounds for our online learning algorithm (which is to be given later in Section 4). The roadmap of the theoretical analysis is depicted in Figure 1.

### 3.1. Regret of Nonstationarity

In this part, we shall introduce our transient analysis approach, which leads to a theoretic upper bound for $R_1(L)$. As we shall see later in Section 4, transient analysis results are also essential for bounding the bias $B_k$ and variance $V_k$ of the gradient estimators in specific online SGD algorithms.

**A crude $O(L)$ bound.** Roughly speaking, since the parameters $\mu, p$ and functions $\lambda(\cdot), c(\cdot)$ are all bounded, the regret $R_1(L)$ is in the same order as the transient bias of the waiting time process, i.e.

$$R_1(L) \approx \sum_{k=1}^{L} O \left( \sum_{n=1}^{D_k} (\mathbb{E}[W_n(\mu_k, p_k)] - \mathbb{E}[W^\infty(\mu_k, p_k)]) \right).$$
Here we use $W_\infty(\mu, p)$ to denote the steady-state waiting time of the $GI/GI/1$ queue with parameter $(\mu, p)$, for any $(\mu, p) \in \mathcal{B}$. Under the uniform stability condition (Assumption 1), it is not difficult to show that there exist positive constants $\gamma > 0$ and $K > 0$ independent of $k$ and $(\mu_k, p_k)$ such that

$$\left| E[W_n^k] - E[W_\infty(\mu_k, p_k)] \right| \leq e^{-\gamma n} K.$$  

Then, as a direct consequence, we have

$$\sum_{n=1}^{D_k} (E[W_n(\mu_k, p_k)] - E[W_\infty(\mu_k, p_k)]) \leq \frac{K}{1 - e^{-\gamma}} \Rightarrow R_1(L) = O(L).$$

An analogue of the above $O(L)$ bound for the regret of nonstationarity in an inventory model is given by Huh et al. (2009) (Lemma 11).

**An improved $o(L)$ bound.** In the rest of this subsection, we will conduct a more delicate analysis on the transient performance of the queueing system, and our analysis will yield a (tighter) sub-linear bound $R_1(L) = o(L)$ (of which the exact order depends on the concrete algorithm, as we shall see later).

**Theorem 1. (Regret of nonstationarity)** Suppose that Assumptions 1 and 2 hold. In addition, assume that the following conditions are satisfied:

(a) $D_k \geq 2 \log(k)/\gamma$;

(b) There exists constants $K_2 > 0$ and $0 < \alpha \leq 1$, such that $E[\|x_k - x_{k+1}\|^2] \leq K_2 k^{-2\alpha}$,
for some $\alpha > 0$ and the constant $\gamma$ defined in Lemma 2. Then, there exists a positive constant $K > 0$ such that

$$R_{1,k} \leq K \cdot k^{-\alpha}, \quad k \geq 1 \quad \text{and} \quad R_{1}(L) \leq K \sum_{k=1}^{L} k^{-\alpha}, \quad L \geq 1. \quad (9)$$

**Remark 4.** As will become clear later in Section 4, we will obtain a bound $R_{1}(L) = O(\log(L))$ for Algorithm 1 by validating Condition (b) in Theorem 1 with $\alpha = 1$, which is much tighter than the crude $O(L)$ bound. This $O(\log(L))$ bound for $R_{1}(L)$ is critical to achieving an overall logarithmic regret bound in the total number of served customers. An explicit expression of constant $K$ is given in (EC.4).

### 3.1.1. Roadmap of the proof of Theorem 1

Our point of departure in proving Theorem 1 is to decompose $R_{1,k}$ into four terms. Specifically, we write

$$R_{1,k} = \rho_{k} - f(\mu_{k}, p_{k})E[T_{k}]$$

$$= E \left[ \sum_{n=1}^{Q_{k}} (h_{0}(W_{n}^{k} + S_{n}^{k}) - p_{k-1}) + \sum_{n=Q_{k}+1}^{D_{k}} (h_{0}(W_{n}^{k} + S_{n}^{k}) - p_{k}) + c(\mu_{k})T_{k} - f(\mu_{k}, p_{k})T_{k} \right]$$

$$= h_{0} \sum_{n=1}^{Q_{k}} \left( E[W_{n}^{k}] - E[W_{\infty}(\mu_{k}, p_{k-1})] \right) + h_{0} \sum_{n=Q_{k}+1}^{D_{k}} \left( E[W_{n}^{k}] - E[W_{\infty}(\mu_{k}, p_{k})] \right)$$

$$+ h_{0} Q_{k} \left( E[W_{\infty}(\mu_{k}, p_{k-1})] - E[W_{\infty}(\mu_{k}, p_{k})] \right)$$

$$+ \left( D_{k} - \lambda_{k} E[T_{k}] \right) \left( h_{0} E[W_{\infty}(\mu_{k}, p_{k})] + \frac{h_{0}}{\mu_{k}} - p_{k} \right) + E[Q_{k} (p_{k} - p_{k-1})] \right) .$$

To show $R_{1,k} = O(k^{-\alpha})$ is equivalent to showing that $E[I_{i}] = O(k^{-\alpha})$ for $i = 1, 2, 3, 4$. The terms $I_{1}$ and $I_{2}$ depend on (i) the rate at which $W_{n}^{k}$, the waiting time in a $GI/GI/1$ queue, converges to its steady state $W_{\infty}$ and (ii) the difference between the initial waiting times and the steady-state waiting times; The term $I_{3}$ relies on (iii) the smoothness of the steady-state waiting time with respect to parameters $(\mu, p)$. To prove Theorem 1, we will establish uniform bounds for (i) in Lemma 2, and for (iii) in Lemma 3. We then show that (ii) can be bounded using the bounds for (i) and (iii). In this way, we obtain the bounds for $I_{1}, I_{2}$ and $I_{3}$. Finally, the term $I_{4}$ is controlled because $E[Q_{k}]$ is uniformly bounded.
We first establish the rate at which waiting times converge to their steady state distributions. For two given sequences \( V_n \) and \( U_n \), we say two GI/GI/1 queues with the same parameter \((\mu, p) \in B\) are synchronously coupled if their waiting times \( W_n^1 \) and \( W_n^2 \) satisfy
\[
W_n^i = \left( W_{n-1}^i + \frac{V_n}{\mu} - \frac{U_n}{\lambda(p)} \right)^+, \quad \text{for } i = 1, 2, \text{ and } n \geq 1,
\]
i.e. the two systems share the same sequences of service and interarrival times. In the following analysis, we introduce two lemmas to establish uniform bounds for the convergence rate. Their proofs are given in Section EC.1.

**Lemma 2.** *(Exponential loss of memory of initial state)* Suppose two GI/GI/1 queues with parameter \((\mu, p) \in B\) are synchronously coupled. If the initial waiting times \((W_0^1, W_0^2)\) are coupled in the same probability space, then there exist two positive constants \(\gamma > 0 \text{ and } \theta > 0\), independent of \((\mu, p)\), such that for any \(m \geq 1\), conditional on \((W_0^1, W_0^2)\),
\[
\mathbb{E} \left[ |W_n^1 - W_n^2|^m \mid (W_0^1, W_0^2) \right] \leq e^{-\gamma n}(2 + e^{\theta W_0^1} + e^{\theta W_0^2})|W_0^1 - W_0^2|^m, 
\]
with \(\gamma = \gamma_0 \cdot \min(1/{\bar{\mu}} , 1/\lambda(p))\), where the constants \(\gamma_0\) and \(\theta\) are defined in Assumption 2.

**Bounding \(I_1\) and \(I_2\).** In order to bound \(I_1\) and \(I_2\) in each cycle \(k\), we apply the synchronous coupling twice. In particular, we first couple the initial waiting time \(W_0^k\) with a steady-state waiting time \(\tilde{W}_0^k\) with parameter \((\mu_k, p_{k-1})\) and let
\[
\tilde{W}_n^k = \left( \tilde{W}_{n-1}^k + \frac{V_n}{\mu_k} - \frac{U_n}{\lambda^{(k)}(p)} \right)^+, 1 \leq n \leq \tilde{Q}_k,
\]
so that \(\tilde{W}_n^k\) is again the stationary waiting time, i.e. \(\tilde{W}_n^k \overset{d}{=} W^\infty(\mu_k, p_{k-1})\) for all \(1 \leq n \leq \tilde{Q}_k\).

In a similar way, for \(n \geq \tilde{Q}_k\), we couple \(W_n^k\) synchronously with a stationary waiting time sequence \(\tilde{W}_n^k\) of parameter \((\mu_k, p_k)\). As a direct corollary of Lemma 2, we show that \(I_1\) and \(I_2\) are bounded by the difference between the initial states \((W_0^k, \tilde{W}_0^k)\) and \((W_{\tilde{Q}_k}^k, \tilde{W}_{\tilde{Q}_k}^k)\), respectively.

**Corollary 1.** Under conditions in Lemma 2, there exists a constant \(A > 0\) independent of \(k\) and \((\mu_k, p_k)\), such that for all \(k\), conditional on \(Q_k\), \((W_0^k, \tilde{W}_0^k)\) and \((W_{\tilde{Q}_k}^k, \tilde{W}_{\tilde{Q}_k}^k)\),
\[
\left| \sum_{n=1}^{Q_k} \mathbb{E} \left[ W_n^k - \tilde{W}_n^k \mid W_0^k, \tilde{W}_0^k \right] \right| \leq A \mathbb{E} \left[ \left| W_0^k - \tilde{W}_0^k \right|^2 \right]^{1/2},
\]
\[
\left| \sum_{n=Q_k+1}^{D_k} \mathbb{E} \left[ W_n^k - \tilde{W}_n^k \mid W_{\tilde{Q}_k}^k, \tilde{W}_{\tilde{Q}_k}^k \right] \right| \leq A \mathbb{E} \left[ \left| W_{\tilde{Q}_k}^k - \tilde{W}_{\tilde{Q}_k}^k \right|^2 \right]^{1/2},
\]
where a closed-form expression of constant \(A\) is given by (EC.2).
Therefore, to show $|I_1| = O(k^{-\alpha})$, it remains to show $\mathbb{E}\left[|W_0^k - \tilde{W}_0^k|^2\right] = O(k^{-2\alpha})$. Note that $\tilde{W}_0^k \overset{d}{=} W_\infty(\mu_k, p_{k-1})$, and $W_0^k = W_{D_{k-1}}^{k-1}$, which, by Lemma 2, is close in distribution to the steady-state waiting time $W_{D_{k-1}}^{k-1} \overset{d}{=} W_\infty(\mu_{k-1}, p_{k-1})$. So essentially we need to couple the two steady-state waiting times $W_\infty(\mu_{k-1}, p_{k-1})$ and $W_\infty(\mu_k, p_{k-1})$ such that

$$
\mathbb{E}\left[|W_\infty(\mu_k, p_{k-1}) - W_\infty(\mu_k, p_k)|^2\right] = O(|\mu_k - \mu_{k-1}|^2) = O(k^{-2\alpha}),$
$$
where the last equality follows from Condition (b) in Theorem 1.

**Bounding $I_3$.** Controlling $\mathbb{E}[I_3]$ also involves bounding the difference of two steady-state waiting times with different model parameters $(p_i, \mu_i)$. Hence, in the following Lemma 3, we establish a uniform higher-order smoothness result for the steady-state waiting times with respect to the model parameter $(\mu, p)$.

**Lemma 3.** (Smoothness in $\mu$ and $p$) Suppose $(\mu_i, p_i) \in \mathcal{B}$ for $i = 1, 2$. Let $W_\infty(\mu_i, p_i)$ be the steady-state waiting time of the GI/GI/1 queue under parameter $(\mu_i, p_i)$, respectively. Then, the steady-state waiting times $(W_\infty(\mu_1, p_1), W_\infty(\mu_2, p_2))$ can be coupled such that, there exists a constant $B > 0$ independent of $(\mu, p)$ satisfying that, for all $1 \leq m \leq 4$,

$$
\mathbb{E}[|W_\infty(\mu_1, p_1) - W_\infty(\mu_2, p_2)|^m] \leq B (|\mu_1 - \mu_2|^m + |p_1 - p_2|^m),
$$

where a closed-form expression of constant $B$ is given in (EC.3).

We adopt a “coupling from the past” (CFTP) approach in the proof of Lemma 3 (see Section EC.1). Roughly speaking, CFTP is a synchronous coupling starting from infinite past. In the proof of Lemma 3, we shall explicitly explain how to construct the CFTP. Given CFTP, then we can couple $(W_0^k, \tilde{W}_0^k)$ and $(W_{Q_k}^k, \tilde{W}_{Q_k}^k)$ for $k \geq 1$ inductively. In detail, for $k = 1$, we just couple $W_0^1$ and $\tilde{W}_0^1$ arbitrarily, and then couple $\tilde{W}_{Q_k}^1$ and $\tilde{W}_{Q_k}^1$ using CFTP. As $W_{Q_k}^1$ and $\tilde{W}_{Q_k}^1$ are synchronously coupled, $W_{Q_k}^1$ is coupled with $W_{Q_k}^1$ accordingly. Then, by induction, conditional on the first $k - 1$ cycles, in cycle $k$, we are able to couple $\tilde{W}_0^k$ with $\tilde{W}_{D_{k-1}}^{k-1}$ again using CFTP and then with $W_0^k = W_{D_{k-1}}^{k-1}$ accordingly. Thus we can complete the coupling between the waiting time sequences $W_n^k$, $\tilde{W}_n^k$ and $\tilde{W}_n^k$ for all $k \geq 1$.

**Finishing the proof of Theorem 1** It suffices to show that $|I_i| = O(k^{-\alpha})$, for $i = 1, 2, 3, 4$. First, by Corollary 1, we have

$$
|I_1| \leq \mathbb{A} \mathbb{E}[|W_0^k - \tilde{W}_0^k|^2]^{1/2} \leq A \left(2 \mathbb{E}[|W_{D_{k-1}}^{k-1} - \tilde{W}_{D_{k-1}}^{k-1}|^2] + 2 \mathbb{E}[|\tilde{W}_{D_{k-1}}^{k-1} - \tilde{W}_0^k|^2] \right)^{1/2}.
$$
On the one hand, by Lemma 2, the uniform bound in Lemma 1 and our choice of $D_{k-1}$
\[ \mathbb{E}[|W_{D_{k-1}}^{k-1} - \bar{W}_{D_{k-1}}^{k-1}|^2] \leq e^{-\gamma D_{k-1}} A \mathbb{E}[|W_0^{k-1} - \bar{W}_0^{k-1}|^4]^{1/2} = O(k^{-2}). \]

In addition, by Lemma 3, $\mathbb{E}[|\bar{W}_{D_{k-1}}^{k-1} - \bar{W}_0^{k-1}|^2] = O(k^{-2\alpha})$. Now we can conclude that $|I_1| = O(k^{-\alpha})$, and $|I_2| = O(k^{-\alpha})$ follows a similar argument. As a direct consequence of Lemma 3, we immediately have $|I_3| = O(k^{-\alpha})$. Regarding $I_4$, we need to control the error $|D_k - \lambda_k \mathbb{E}[T_k]|$. Because $D_k / \mathbb{E}[T_k]$ is close to $\lambda_k$ in steady state, we have $|I_4| = O(k^{-\alpha})$ by analyzing the transient bias similarly as we did for $I_1$ and $I_2$.

Remark 5. The complete proof of Theorem 1 is given in Section EC.1.5 of the e-companion. We advocate that Theorem 1 may apply to other queueing models (its scope is beyond the GI/GI/1 queue), as long as one can verify two conditions for the designated model: (i) uniform boundedness for the rate of convergence to the steady state, i.e. Lemma 2 and (ii) smoothness of the stationary distributions in the control variables, i.e. Lemma 3.

3.2. Regret of Suboptimality

To bound the regret of suboptimality $R_2(L)$, we need to control the rate at which $x_k$ converges to $x^*$. This depends largely on the effectiveness of the estimator $H_k$ for $\nabla f(x_k)$. In our algorithm, such effectiveness is measured by the bias $B_k$ and variance $V_k$. The following result shows that, if $B_k$ and $V_k$ can be appropriately bounded, then, $x_k$ will converge to $x^*$ rapidly and hence $R_2(L)$ is also bounded.

Theorem 2. (Regret of suboptimality) Suppose Assumptions 3 holds. Let $0 < \beta \leq 1$. Recall that $\eta_k$ is the step size in SGD iteration. Then, if there exists a constant $K_3 \geq 1$ such that the following conditions hold for all $k$,
\[ (a) \quad (1 + \frac{1}{k-1})^\beta \leq 1 + \frac{K_0}{2} \eta_k, \]
\[ (b) \quad B_k \leq \frac{K_0}{8} k^{-\beta}, \]
\[ (c) \quad \eta_k V_k \leq K_3 k^{-\beta}, \]
then, for all $k \geq 1$ and $C \geq \max\{\|x_1 - x^*\|^2, 8K_3/K_0\}$
\[ \mathbb{E}[\|x_k - x^*\|^2] \leq C \cdot (k - 1)^{-\beta}, \tag{10} \]

and as a consequence,
\[ R_2(L) \leq CK_1 \sum_{k=1}^{L} \left( \frac{D_k}{\lambda(p)} + M \right) k^{-\beta} = O \left( \sum_{k=1}^{L} D_k k^{-\beta} \right). \tag{11} \]
Remark 6 (Optimal choice for parameter $D_k$). The above expression (11) indicates a trade-off in the choice of the parameter $D_k$. On the one hand, increasing the sample size $D_k$ reduces the bias $B_k$ for the gradient estimator, and hence leads to smaller value of $k^{-\beta}$. On the other hand, a larger $D_k$ makes the system operate under a sub-optimal decision for a longer time. To this end, one may choose an optimal $D_k$ that minimizes the order of the regret as in (11).

Our proof of Theorem 2 follows an inductive approach as used in Broadie et al. (2011). Let $b_k \equiv \mathbb{E}[\|x_k - x^*\|^2]$. According the SGD iteration $x_{k+1} = \Pi_B(x_k - \eta_k H_k)$, we have

$$
\mathbb{E}[\|x_{k+1} - x^*\|^2|x_k] \leq \mathbb{E}[\|x_k - \eta_k H_k - x^*\|^2|x_k] = \|x_k - x^*\|^2 - 2\eta_k \mathbb{E}[H_k|x_k](x_k - x^*) + \eta_k^2 \mathbb{E}[H_k^2|x_k].
$$

Then, by Assumption 3 and the definition of $B_k$, $V_k$ by (4), we derive the following recursive inequality for $b_k$:

$$
b_{k+1} \leq (1 - K_0 \eta_k + \eta_k B_k)b_k + \eta_k B_k + \eta_k^2 V_k, \quad k \geq 1,
$$

and prove (10) by induction. The full proof is given in Section EC.1.7 of the e-companion.

In Section 4, we apply Theorem 2 to treat our online SGD algorithm (Algorithm 1) by verifying that Conditions (a)–(c) are satisfied. Because in Theorem 2, Conditions (a)–(c) are stated explicitly in terms of the step size $\eta_k$, bias $B_k$ and variance $V_k$ of the gradient estimator, these conditions may serve as useful building blocks for the design and analysis of online learning algorithms in other queueing models.

4. An Online Learning Algorithm of the $GI/GI/1$ Queue

In this section, we provide a concrete SGD-based online learning algorithm that solves the optimal pricing and capacity sizing problem (1) for a $GI/GI/1$ queueing system. We show that gradient $\nabla f(\mu, p)$ can be estimated “directly” from past experience (i.e., data of delay and busy times generated under present policy). Applying the regret analysis developed in Section 3, we provide a theoretic upper bound for the overall regret in Theorem 3.

4.1. A Gradient Estimator

Following the algorithm framework outlined in Section 2.2, we now develop a detailed gradient estimator $H_k$. Regarding the objective function in (5), it suffices to construct estimators for the partial derivatives

$$
\frac{\partial}{\partial \mu} \mathbb{E}[W_\infty(p, \mu)] \quad \text{and} \quad \frac{\partial}{\partial p} \mathbb{E}[W_\infty(p, \mu)].
$$

(12)
Following the infinitesimal perturbation analysis (IPA) approach (see, for example, Glasserman (1992)), we next show that the partial derivatives in (12) can be expressed in terms of the steady-state distributions $W_\infty(p,\mu)$ and $X_\infty(p,\mu)$ of the delay process $W_n$ and observed busy period process $X_n$, of which the dynamics are characterized by (6)–(7).

**Lemma 4.** If a GI/GI/1 queue is stable under parameter $(\mu, p)$, we have

$$\begin{align*}
\frac{\partial}{\partial p} f(\mu, p) &= -\lambda(p) - p\lambda'(p) + h_0\lambda'(p) \left( \mathbb{E}[W_\infty(\mu, p)] + \mathbb{E}[X_\infty(\mu, p)] + \frac{1}{\mu} \right), \\
\frac{\partial}{\partial \mu} f(\mu, p) &= c'(\mu) - h_0\frac{\lambda'(p)}{\lambda(p)} \left( \mathbb{E}[W_\infty(\mu, p)] + \mathbb{E}[X_\infty(\mu, p)] + \frac{1}{\mu} \right). 
\end{align*}$$

(13)

**Proof of Lemma 4.** To prove Equation (13), it suffices to work with the partial derivatives of the steady-state expectation $\mathbb{E}[W_\infty(\mu, p)]$. We follow the IPA analysis in Glasserman (1992) and Chen (2014).

Given $(\mu, p)$, we define $r(p) = 1/\lambda(p)$ and rewrite the recursion (6) as

$$W_n(\mu, p) = \left( W_{n-1}(\mu, p) + \frac{V_n}{\mu} - rU_n \right)^+.$$ 

Define the derivative process $Y_n \equiv \frac{\partial}{\partial r}W_n(\mu, p)$, then by chain rule, we have

$$Y_n = \frac{\partial}{\partial r}W_n(\mu, p) = \frac{\partial}{\partial r} \left( W_{n-1}(\mu, p) + \frac{V_n}{\mu} - rU_n \right)^+ = \begin{cases} 
\frac{\partial}{\partial r}W_{n-1} - U_n = Y_{n-1} - U_n & \text{if } W_n > 0; \\
0 & \text{if } W_n = 0.
\end{cases}$$

and obtain a recursion $Y_n = (Y_{n-1} - U_n) 1_{\{W_n > 0\}}$. Let $\tilde{Y}_n \equiv -Y_n/\lambda(p)$. Then, it is straightforward to see that $\tilde{Y}_n$ follows the recursion given in (7) as the observed busy period $X_n$, i.e.

$$\tilde{Y}_n = \left( \tilde{Y}_{n-1} + \frac{U_n}{\lambda(p)} \right) 1(W_n > 0).$$

Under the assumption that the queueing system is stable, the limit $\tilde{Y}_\infty$ should be equal in distribution with $X_\infty$. Therefore, we formally derive

$$\frac{\partial}{\partial r} \mathbb{E}[W_\infty(\mu, p)] = \mathbb{E}[Y_\infty] = -\lambda(p)\mathbb{E}[\tilde{Y}_\infty] = -\lambda(p)\mathbb{E}[X_\infty(\mu, p)].$$

(14)

The above heuristics can be made rigorous by verifying exchanges of limits. We refer the readers to Glasserman (1992) for detailed treatments. Given (14), we can derive the partial derivative of the steady-state waiting time with respect to price $p$ as

$$\frac{\partial}{\partial p} \mathbb{E}[W_\infty(\mu, p)] = \frac{\partial}{\partial r} \mathbb{E}[W_\infty(\mu, p)] \frac{\partial r(p)}{\partial p} = -\lambda(p)\mathbb{E}[X_\infty(\mu, p)] \cdot \frac{\lambda'(p)}{\lambda(p)^2} = \mathbb{E}[X_\infty(\mu, p)] \frac{\lambda'(p)}{\lambda(p)}.$$
Now we turn to \( \frac{\partial}{\partial \mu} \mathbb{E}[W_\infty(\mu, p)] \). Let \( \hat{W}_n = \mu W_n(\mu, p) \), it is easy to check that \( \hat{W}_n = \left( \hat{W}_n + V_n - \mu U_n / \lambda(p) \right)^+ \). Then, following a similar derivation as we did in (14), we have
\[
\frac{\partial}{\partial \mu} \mathbb{E}[\hat{W}_\infty(\mu, p)] = -\mathbb{E}[X_\infty(\mu, p)].
\]

Therefore,
\[
-\mathbb{E}[X_\infty(\mu, p)] = \frac{\partial}{\partial \mu} \mathbb{E}[\hat{W}_\infty(\mu, p)] = \frac{\partial}{\partial \mu} \mathbb{E}[\mu W_\infty(\mu, p)] = \mu \frac{\partial}{\partial \mu} \mathbb{E}[W_\infty(\mu, p)] + \mathbb{E}[W_\infty(\mu, p)],
\]
and hence, \( \partial \mathbb{E}[W_\infty(\mu, p)] / \partial \mu = -\left( \mathbb{E}[X_\infty(\mu, p)] + \mathbb{E}[W_\infty(\mu, p)] \right) / \mu \).

Finally, plugging the expressions of the two partial derivatives into \( \nabla f \) yields (13). \( \square \)

### 4.2. An Online Learning Algorithm for GI/GI/1

Utilizing results in Lemma 4, we are ready to design an online SGD algorithm by estimating the terms \( \mathbb{E}[W_\infty(\mu, p)] \) and \( \mathbb{E}[X_\infty(\mu, p)] \) in the partial derivatives (13) by the finite-sample averages of \( W_n^k \) and \( X_n^k \) observed in each cycle \( k \). The formal description of the algorithm is given in Algorithm 1.

---

**Algorithm 1:** Online SGD Algorithm for GI/GI/1 Queues

**Input:** number of cycles \( L \);

parameter \( 0 < \xi < 1 \), \( D_k, \eta_k \) for \( k = 1, 2, ..., L \);

initial value \( x_1 = (\mu_1, p_1) \);

for \( n = 1, 2, ..., D_k \) do

operate the system under \( x_k = (\mu_k, p_k) \) until \( D_k \) customers enter service;

observe \( (W_n^k, X_n^k) \) for \( n = 1, 2, ..., D_k \);

randomly draw \( Z \in \{1, 2\} \);

if \( Z = 1 \) then

\[
\begin{align*}
 h &\leftarrow -\lambda(p_k) - p_k \lambda'(p_k) + h_0 \lambda'(p_k) \left( \frac{1}{D_k(1-\xi)} \sum_{n>\xi D_k}^{D_k} (X_n^k + W_n^k) + \frac{1}{\mu_k} \right); \\
 H_k &\leftarrow (h, 0);
\end{align*}
\]

else

\[
\begin{align*}
 h &\leftarrow c'(\mu_k) - h_0 \frac{\lambda(p)}{\mu_k} \left( \frac{1}{D_k(1-\xi)} \sum_{n>\xi D_k}^{D_k} (X_n^k + W_n^k) + \frac{1}{\mu_k} \right); \\
 H_k &\leftarrow (0, h);
\end{align*}
\]

end

update: \( x_{k+1} = \Pi_B(x_k - \eta_k H_k) \);

end
Remark 7 (Parameter optimization for Algorithm 1). It now remains to select the optimal parameters $D_k$ and $\eta_k$ in Algorithm 1 so as to achieve the minimal asymptotic regret. In general, one can first carry out an analysis on the specific gradient estimator used in the algorithm to obtain bounds on $B_k$ and $V_k$ in terms of the algorithm parameters. Then, applying the regret analysis framework introduced in Section 3, one may obtain the optimal parameters using the expression of the regret bound, which depends on algorithm parameters, $B_k$ and $V_k$, i.e. (9) and (11). In the case of Algorithm 1, we directly show as below that, appropriate choices of $D_k$ and $\eta_k$ (along with the corresponding $B_k$ and $V_k$), will satisfy the conditions in Theorem 1 with $\alpha = 1$ and those in Theorem 2 with the largest possible value of $\beta$ (i.e., $\beta = 1$). As a consequence, we can prove that Algorithm 1 has a regret bound of $O((\log(M_L)^2)$ with $M_k \equiv \sum_{k=1}^{L} D_k$ being the cumulative number of customers served by cycle $L$.

Theorem 3. (Regret Bound for Algorithm 1)
Suppose Assumptions 1 to 3 hold. If $D_k = O(\log(k))$ and $\eta_k = 2k^{-1}/K_0$, $k = 1, 2, ..., L$, then

(i) There exists a positive constant $K_3 > 0$ such that

$$B_k \leq \frac{K_0}{8k} \quad \text{and} \quad \eta_k V_k \leq \frac{K_3}{k}.$$  

(ii) There exists a positive constant $K_4 = K_3/K_0$ such that

$$\mathbb{E}[\|x_k - x_{k+1}\|^2] \leq K_4 k^{-2}.$$  

(iii) As a consequence of (i) and (ii), the regret for Algorithm 1

$$R(L) \leq K_{alg} \log(M_L)^2 = O(\log(M_L)^2).$$  

Remark 8 (On the logarithmic regret bound (16)). Below we provide some additional discussions on the regret bound (16):

(i) On the constant $K_{alg}$. The explicit expression for the constant $K_{alg}$, although complicated, is given by (EC.8). It involves error bound corresponding to the transient behavior of the queueing system, the bias and variance of the gradient estimator, moment bounds on the queue length and other model parameters. One can verify that $K_{alg}$ is increasing in the convergence rate coefficient $\gamma$ and in the moment bounds of the queue length $M$. In Section 5, we conduct numerical experiments to test how $K_{alg}$ depends on model parameters. Explicit formulas for parameters $D_k$ and $K_3$ are given in (EC.6) and (EC.7).
(ii) **On the first logarithmic term.** Consider an SGD algorithm in that an unbiased gradient estimator $H_k$ with a bounded variance can be evaluated using a single data point (i.e. $B_k = 0$, $V_k = O(1)$), it has been proved the scaled error $k^{-1/2}(x_k - x^*)$ converges in distribution to a non-zero random variable (Theorem 2.1 in Chapter 10 of Kushner and Yin (2003)). Hence, the optimal convergence rate for $\|x_k - x^*\|^2$ that any SGD-based algorithm can achieve is at best $O(k^{-1})$ (yielding a cumulative regret of order $O(\log(k))$), which is exactly the rate of convergence established by our online algorithm (see the proof of Theorem 3). In this sense, our algorithm indeed achieves a minimal convergence rate, except that our gradient estimator is obtained using an increasing number of data points. This is because, in presence of the nonstationary error of the queueing system, it is not possible to obtain an unbiased estimator of the steady-state gradient using a constant number of data points in each cycle.

(iii) **On the second logarithmic term.** In order to control the regret of nonstationarity, the queueing system need to be operated in each cycle for a duration $O(\log(k))$. Because the queueing performance converges to its steady state exponentially fast, this inevitably introduces an extra logarithmic term in our regret bound (which explains the “square” in $\log(M_L)^2$). The question that remains open is whether this $O(\log(M_L)^2)$ bound is optimal. We conjecture that the answer is yes but admit that a rigorous treatment of a lower regret bound can be quite challenging. For example, establishing a lower regret bound requires a lower bound on the convergence rate of a $GI/GI/1$ queue, which itself is an open question. We leave this question to future research.

**Remark 9 (Controlling the length of cycle $k$).** We use $D_k$ (the number of customers served in cycle $k$), instead of the clock time $T_k$, to control and measure the regret bound. The benefit of using $D_k$ (rather $T_k$) as the cycle length is that it facilitates the technical analysis, because $D_k$ is directly related with the number of samples used to estimate our gradient estimator. In fact, using $D_k$ instead of $T_k$ has no bearing on the order of the regret bound. To see this, note that the arrival rate is assumed to fall into a compact set $[\lambda(\bar{p}), \lambda(\bar{p})]$. Therefore, since $T_L$ be the total units of clock time elapsed after cycle $L$, we have $M_L/\lambda(p) \leq \mathbb{E}[T_L] \leq M_L/\lambda(\bar{p})$ for all $L$. 
5. Numerical Experiments

To confirm the practical effectiveness of our online learning method, we conduct numerical experiments to visualize the algorithm convergence, benchmark the outcomes with known exact optimal solutions, estimate the true regret and compare it to the theoretical upper bounds. Our base example is an $M/M/1$ queue, having Poisson arrivals with rate $\lambda(p)$, and exponential service times with rate $\mu$. In our optimization, we consider a commonly used logistic demand function (Besbes and Zeevi (2015))

$$\lambda(p) = M_0 \frac{\exp(a - p)}{1 + \exp(a - p)},$$

and a convex cost function for the service rate

$$c(\mu) = c_0 \mu^2.$$  

See the top left panel of Figure 2 for $\lambda(p)$ in (17). In particular, the optimal pricing and staffing problem in (1) now becomes

$$\max_{\mu,p} \left\{ p\lambda(p) - c_0 \mu^2 - h_0 \frac{\lambda(p)/\mu}{1 - \lambda(p)/\mu} \right\}. 

\text{(19)}$$

In light of the closed-form steady-state formulas of the $M/M/1$ queue, we can analytically obtain the exact values of the optimal solutions $(p^*, \mu^*)$ and the corresponding objective value $f(\mu^*, p^*)$, with which we are able to benchmark the solutions from our online optimization algorithm.

We first consider two one-dimensional online optimization problems in Section 5.1: online optimal pricing and online optimal staffing. We next treat the two-dimensional pricing and staffing problem in Section 5.2. In Section 5.3, we compare our results to previously established asymptotic heavy-traffic solutions in Lee and Ward (2014) as the system scale increases. Additional numerical experiments are provided in the e-companion: In Section EC.2.1 we show that our algorithm remains effective without the uniform stability condition. In Sections EC.2.2 and EC.2.3 we consider the more general $GI/GI/1$ examples with phase-type and lognormal service time distributions.
5.1. One-Dimensional Online Optimizations

5.1.1. Online optimal pricing with a fixed service capacity

Motivated by revenue management problems in revenue generating service system, our first example focuses on the one-dimensional optimization of price $p$ with service rate $\mu = \mu_0$ held fixed. In this case we can simply omit the term $c_0\mu^2$ in (19). Fixing the other model parameters as $a = 4.1, M_0 = 10, h_0 = 1$ and $\mu_0 = 10$, we first obtain the exact optimal price $p^* = 3.531$ (top right panel of Figure 2). According to Algorithm 1 and Theorem 3, we set the step size $\eta_k = 1/k$ and cycle length $D_k = 10 + 10 \log(k)$, with initial price $p_0 = 6.5$. In Figure 2, we give the sample paths of the gradient $H_k$ and price $p_k$ as functions of the number of cycles $k$, and the mean regret (estimated by averaging 500 independent sample paths) as a function of the cumulative number of service completions $M_L$. We observe that the pricing decision $p_k$ quickly converges to the optimal value $p^*$, and the regret grows as a logarithmic function of $M_L$. In particular, a simple linear regression for the pair $(\sqrt{R(M_L)}, \log(M_L))$ (bottom right panel) verifies our regret bound given in Theorem 3.

5.1.2. Online optimal staffing problem with exogenous arrival rate

Motivated by conventional service systems where customers are served based on good wills (e.g., hospitals), we next solve an online optimal staffing problem, with the objective of minimizing the combination of the steady-state queue length (or equivalently the delay) and the staffing cost, with the arrival rate held fixed. Namely, we omit the term $p\lambda(p)$ in (19). Fixing $\lambda = \lambda_0 = 6.385, h_0 = 1$, and $c_0 = 0.2$, we obtain the exact optimal service capacity $\mu^* = 8.342$ (top right panel of Figure 3). Also by Algorithm 1 and Theorem 3, we set the step size $\eta_k = 0.4k^{-1}$ and cycle length $D_k = 10 + 10 \log(k)$ with initial service rate $\mu_0 = 10$. In Figure 3, we again give sample paths of the gradient $H_k$ and service capacity $\mu_k$, and estimation of the regret. As the number of cycles $k$ increases, our stage-$k$ staffing decision $\mu_k$ quickly converges to $\mu^*$ (bottom right panel) and the regret also grows as a logarithmic function of $M_L$ (bottom left panel).

5.2. Joint Pricing and Staffing Problem

We next consider the joint staffing and pricing problem having the objective function in (19), with the logistic demand function in (17) and parameters $a = 4.1, M_0 = 10, h_0 = 1$ and $c_0 = 0.1$. The optimal price $p^* = 4.02$ and service rate $\mu^* = 7.10$ are given as benchmarks (top right panel in Figure 4). In Figure 4, we show that $\mu_k$ and $p_k$ converge quickly to their
corresponding optimal target levels $\mu^*$ and $p^*$. And similar to the one-dimensional cases, the regret grows as a logarithmic function of $M_L$ (bottom left panel). We next use this example to test the performance of the online learning algorithm without the requirement of uniform stability (i.e., when Condition (c) of Assumption 1 is violated). Specifically, we pick the initial pricing and staffing policies $(p_0, \mu_0)$ such that the beginning traffic intensity is $\rho_0 = \lambda(p_0)/\mu_0 > 1$. Our simulation results show that the algorithm remains effective with the $GI/GI/1$ queue quickly becoming stable. See Section EC.2.1 for details.
Figure 3  Online optimal staffing for an M/M/1 queue with fixed price with $\lambda_0 = 6.385, M_0 = 10, \eta_k = 0.4k^{-1}$ and $D_k = 10 + 10\log(k)$: (i) staffing cost (top left); (ii) cost function (top right); (iii) sample path of gradient (middle left); (iv) sample path of service capacity (middle right); (v) estimated regret (bottom left); (vi) square root of regret versus logarithmic of served customers, with $c = 2.76, d = -8.68$ (bottom right).

5.3. Optimal Solutions as the Scale Increases

It is a common consensus that non-Markovian queues, despite their theoretical and practical importance, can be difficult to analyze, because analytic performance formulas are hardly available. A predominant treatment to queues with non-exponential service and interarrival times is to resort to their heavy-traffic limits, which are much more tractable.
Figure 4 Joint pricing and staffing for an $M/M/1$ queue with $p_0 = 7.5$, $\mu_0 = 12$, $\eta_k = 1/k$ and $D_k = 10 + 10 \log(k)$: (i) demand function (top left); (ii) revenue function (top right); (iii) sample path of gradient (middle left); (iv) sample path of decision parameters (middle right); (v) estimated regret (bottom left); (vi) square root of regret versus logarithmic of served customers, with $c = 0.186$, $d = 5.17$ (bottom right).

For example, one can construct a sequence of $GI/GI/1$ models indexed by $n$, where the $n^{th}$ model has scaled arrival rate $\lambda_n(p) = n\lambda(p)$ and service rate $\mu_n = n\mu$, so that both $\lambda_n$ and $\mu_n$ grow to $\infty$ as $n$ increases. Lee and Ward (2014) develop asymptotic staffing and pricing solutions for the $GI/GI/1$ queue; they show that, as $n \to \infty$, the optimal price $p_n^* \to p_\infty$ and service capacity $\mu_n^*/n \to \mu_\infty$, with $\rho_\infty \equiv \lambda(p_\infty)/\mu_\infty = 1$. 
To make contact with previous heavy-traffic results, we compare the solutions of our online optimization algorithm to the asymptotic solutions in Lee and Ward (2014) by letting the system scale increase. We repeat our experiment in Section 5.2 with the scaling parameter $M_0 \in \{10, 50, 100, 500, 1000, 2000\}$ for the arrival rate function (17). In Figure 5 we plot the optimal price and service rate as $M_0$ increases. In each experiment, for $M_0 = n$, we compute the optimal $p_n$ and $\mu_n$ using their average value in cycles 300–500 of Algorithm 1. Consistent with Lee and Ward (2014), Figure 5 shows that $p_n$, $\mu_n/n$ and $\rho_n$ approach $p_{\infty}$, $\mu_{\infty}$ and $\rho_{\infty} = 1$. On the other hand, when the scale $M_0$ is not very large, the heavy-traffic solutions can become inaccurate. For instance, when $M_0 = 100$ the optimal traffic $\rho_{100}$ is around 0.8, which is not close to 1.

6. Conclusion

In this paper we develop an online learning framework designed for dynamic pricing and staffing in queueing systems. The ingenuity of this approach lies in its online nature, which allows the service provider to continuously obtain improved the pricing and staffing policies by interacting with the environment. The environment here is interpreted as everything beyond the service provider’s knowledge, which is the composition of the random external demand process and the complex internal queueing dynamics. The proposed algorithm organizes the time horizon into successive operational cycles, and prescribes an efficient way to update the service provider’s policy in each cycle using data collected in previous cycles. Data include the number of customer arrivals, waiting times, and the server’s busy times.

A key appeal of the online learning approach is its insensitivity to the scale of the queueing system, as opposed to previous solution approaches, such as heavy-traffic analysis, which require the system to be in large scale (with the arrival and service rate both approaching infinity). In addition, due to its “data-driven” effect, the online learning algorithm is robust to the distributions of service and arrival times; in particular the framework proposed in the present work requires only first-moment information of these random elements. Effectiveness of our online learning algorithm is substantiated by (i) theoretical results including the algorithm convergence and regret analysis, and (ii) engineering confirmation via simulation experiments of a variety of representative $GI/GI/1$ queues. Theoretical analysis of the regret bound in the present paper may shed lights on the design
of efficient RL algorithms (e.g., bounding gradient estimation error and controlling proper learning rate) for more general queueing systems.

There are several venues for future research in this direction. One natural extension would be to extend the method to more general model settings (e.g., queues having customer abandonment and multiple servers), which will make the framework more practical for service systems such as call centers and healthcare. Convergence and regret analysis for online learning methods in other queueing models may be developed based on results
in Section 3; for example, it suffices to devise proper gradient estimators that satisfy the sufficient conditions in Theorems 1 and 2. Another interesting and promising direction is to develop an online learning method without assuming the knowledge of the arrival rate function $\lambda(p)$, where the learner (hereby the service provider), during the interactions with the environment, will have to resolve the tension between obtaining an accurate estimation of the demand function and optimizing returns over time.

References


E-Companion

This e-companion provides supplementary materials to the main paper. In Section EC.1, we provide all the technical proofs omitted from the main paper. In EC.2, we give additional numerical studies. In EC.3, we give further discussions on Assumption 3.

EC.1. Proofs

EC.1.1. Proof of Lemma 1

The proof follows a stochastic ordering argument for GI/GI/1 models. Let $\hat{W}_n^k$, $\hat{X}_n^k$ and $\hat{Q}_n^k$ be the waiting times, observed busy periods, and queue length process in a GI/GI/1 queue with stationary control parameter $\mu_k \equiv \underline{\mu}$ and $p_k \equiv \underline{p}$, and with steady-state initial state, i.e. $\hat{W}_0^1 \overset{d}{=} W_\infty(\mu, p)$, $\hat{X}_0^1 \overset{d}{=} X_\infty(\mu, p)$ and $\hat{Q}_0^1 \overset{d}{=} Q_\infty(\mu, p)$. Let’s call this system the dominating system. Then, for all $k$,

$$\frac{U_n^k}{\lambda_{k-1}} \geq \frac{U_n^k}{\lambda(p)}$$

for $n = 1, 2, ..., Q_k$, and $\frac{U_n^k}{\lambda_k} \geq \frac{U_n^k}{\lambda(p)}$, for $n = Q_k + 1, 2, ..., D_k$,

i.e. the arrival process in the dominating queue is the upper envelope process (UEP) for all possible arrival processes corresponding to any control sequence $(\mu_k, p_k)$. Similarly, the service process in the dominating queue is the lower envelope process (LEP) for all possible service processes corresponding to any control sequence. As a consequence,

$$W_n^k \leq_{st} \hat{W}_n^k, \quad X_n^k \leq_{st} \frac{\lambda(p)}{\lambda(p)} \cdot \hat{X}_n^k, \quad Q_n^k \leq_{st} \hat{Q}_n^k.$$ 

Under Assumption 1, $\mu > \lambda(p)$ and $\mathbb{E}[\exp(\eta V_n/\mu)] < \infty$ for some $\eta > 0$. It is known that, under these conditions, $\mathbb{E}[(\hat{W}_n^k)^m], \mathbb{E}[(\hat{X}_n^k)^m], \mathbb{E}[(\hat{Q}_n^k)^m], \mathbb{E}[\exp(\eta \hat{W}_n^k)], \mathbb{E}[\exp(\eta Q_n^k)]$ are finite for all $m \geq 1$, see Chapter 10 of Asmussen (2003) for more details. Therefore, we choose

$$M = \max_{1 \leq m \leq 4} \left\{ \mathbb{E}[(\hat{W}_n^k)^m], \frac{\lambda(p)^m}{\lambda(p)^m} \mathbb{E}[(\hat{X}_n^k)^m], \mathbb{E}[(\hat{Q}_n^k)^m], \mathbb{E}[\exp(\eta \hat{W}_n^k)], \mathbb{E}[\exp(\eta Q_n^k)] \right\},$$

and this closes our proof. \qed

EC.1.2. Proof of Lemma 2

For $i \in \{1, 2\}$, define a stopping time $\Gamma_i = \min\{n : W_n^i = 0\}$. Let $\gamma = \gamma_0 \cdot \min(1/\bar{\mu}, 1/\lambda(p))$ with $\gamma_0$ as defined in Assumption 2. For a fixed pair of inter-arrival and service time sequences, the consequent waiting time sequence $W_k$ in a single-server queue is monotone
in its initial state $W_0$. Without loss of generality, assume $W_0^1 \geq W_0^2$. Then, $W_n^1 \geq W_n^2$ for all $n \geq 1$ and therefore, $W_1^1 = W_1^2 = 0$. As the two queues are coupled with the same arrival and service time sequences, we will have $W_n^1 = W_n^2$ for all $n \geq \Gamma_1$. Therefore, we can conclude $W_n^1 = W_n^2$ for all $n \geq \max(\Gamma_1, \Gamma_2)$. For $n \leq \max(\Gamma_1, \Gamma_2)$, we have $|W_n^1 - W_n^2| \leq |W_0^1 - W_0^2|$ following Kella and Ramasubramanian (2012).

Now let $\phi_S(t) = \log \mathbb{E}[\exp(tS_1)]$ and $\phi_r(t) = \log \mathbb{E}[\exp(t\tau_1)]$ be the cumulant generating functions of the service and inter-arrival times. Then, $\phi_S(t) = \phi_V(t, \mu)$ and $\phi_r(t) = \phi_U(t, \lambda(p))$. For simplicity of notation, we write $s = 1/\mu$ and $u = 1/\lambda(p)$ as the mean service and inter-arrival times, respectively. Then, under Assumption 2, for $\theta$ as defined in (3), we have

$$\phi_r(-\theta) < -(1-a)u\theta - \gamma, \phi_S(\theta) < (1+a)s\theta - \gamma.$$

For $i \in \{1, 2\}$, define a random walk $R_{n+1}^i = R_n^i + S_n - \tau_n$ with $R_0^i = W_0^i$. By Lindley recursion, $\Gamma_i = \min\{n : R_n^i \leq 0\}$. Then, for any $n \geq 1$,

$$\mathbb{P}(\Gamma_i \leq n) \geq \mathbb{P}\left(\sum_{k=1}^n (S_k - \tau_k) < -W_0^i\right) \geq \mathbb{P}\left(\sum_{k=1}^n \tau_k \geq n(1-a)u, \sum_{k=1}^n S_k \leq n(1+a)s - W_0^i\right),$$

where the second inequality holds as $(1-a)u > (1+a)s$ given that $0 < a < (\mu - \lambda(p))/(\mu + \lambda(p))$ and that $s/u = \lambda(p)/\mu \leq \lambda(p)/\mu$. Therefore,

$$\mathbb{P}(\Gamma_i > n) \leq \mathbb{P}\left(\sum_{k=1}^n \tau_k < n(1-a)u\right) + \mathbb{P}\left(\sum_{k=1}^n S_k > n(1+a)s - W_0^i\right).$$

Following Chebyshev’s Inequality, we have

$$\mathbb{P}\left(\sum_{k=1}^n S_k > n(1+a)s - W_0^i\right) \leq \frac{\mathbb{E}[\exp(\theta \sum_{k=1}^n S_k)]}{\exp(n\theta(1+a)s - \theta W_0^i)} = \exp(n(\phi_S(\theta) - (1+a)s\theta)) \exp(\theta W_0^i) \leq \exp(-n\gamma) \exp(\theta W_0^i).$$

On the other hand, let $Q$ be an exponentially tilted probability measure with respect to $\tau$, such that the likelihood ratio $\frac{dQ}{d\mathbb{P}}(\tau) = \exp(-\theta \tau - \phi_r(-\theta))$. Then,

$$\mathbb{P}\left(\sum_{k=1}^n \tau_k < n(1-a)u\right) = \mathbb{E}^Q\left[\exp\left(\theta \sum_{k=1}^n \tau_k + n\phi_r(-\theta)\right) \mathbf{1}_{\{\sum_{k=1}^n \tau_k < n(1-a)u\}}\right] \leq \exp(n(1-a)u\theta + n\phi_r(-\theta)) = \exp(n((1-a)u\theta + \phi_r(-\theta))) \leq \exp(-n\gamma), i = 1, 2.$$. 
In summary, we have $\mathbb{P}(\Gamma_1 > n) \leq \exp(-n\gamma) (1 + \exp(\theta W_0^i))$. So, we can conclude
\[ E[|W_n^1 - W_n^2|^m] \leq \mathbb{P}(\max(\Gamma_1, \Gamma_2) > n)|W_0^1 - W_0^2|^m \]
\[ \leq e^{-\gamma n}(2 + e^{\theta W_0^1} + e^{\theta W_0^2})|W_0^1 - W_0^2|^m. \]

**EC.1.3. Proof of Corollary 1**

Since $W_n^k$ and $\tilde{W}_n^k$ are synchronously coupled, we have
\[
\sum_{n=1}^{Q_k} (E[W_n^k] - E[\tilde{W}_n^k]) = \sum_{n=1}^{\infty} (E[W_n^k] - E[\tilde{W}_n^k]) 1(n \leq Q_k)
\]
\[ \leq \sum_{n=1}^{\infty} e^{-\gamma n}(2 + e^{\theta W_0^k} + e^{\theta \tilde{W}_0^k})|W_0^k - \tilde{W}_0^k| 1(n \leq Q_k)]
\]
\[ \leq \sum_{n=1}^{\infty} e^{-\gamma n}(E[(e^{\theta W_0^k} + e^{\theta \tilde{W}_0^k})]W_0^k - \tilde{W}_0^k] + 2E[|W_0^k - \tilde{W}_0^k|])
\]
\[ \leq \frac{1}{1 - e^{-\gamma}}(E[(e^{\theta W_0^k} + e^{\theta \tilde{W}_0^k})^2]^{1/2} \cdot E[|W_0^k - \tilde{W}_0^k|^{2}]^{1/2} + 2E[|W_0^k - \tilde{W}_0^k|])
\]

By Lemma 1, we have, as $\theta < \eta/2,$
\[ E[(e^{\eta W_0^k} + e^{\eta \tilde{W}_0^k})^2] \leq 2E[e^{2\eta W_0^k}] + 2E[e^{2\eta \tilde{W}_0^k}] \leq 4M, \text{ besides } E[|W_0^k - \tilde{W}_0^k|] \leq E[|W_0^k - \tilde{W}_0^k|^{2}]^{1/2}.
\]

Therefore,
\[ \left| \sum_{n=1}^{Q_k} E\left[W_n^k - \tilde{W}_n^k \mid W_0^k, \tilde{W}_0^k\right] \right| \leq A E[|W_0^1 - W_0^2|^{2}]^{1/2}, \text{ with } A = \frac{2\sqrt{M} + 2}{1 - e^{-\gamma}}. \quad (EC.2)
\]

The inequality for $W_n^k$ and $\tilde{W}_n^k$ follows the same argument. \qed

**EC.1.4. Proof of Lemma 3**

By the inequality that $(a + b)^m \leq 2^{m-1}(a^m + b^m)$ for $m \geq 1$, we have
\[ E[|W_\infty(\mu_1, p_1) - W_\infty(\mu_2, p_2)|^m] \leq 2^{m-1}(E[|W_\infty(\mu_1, p_1) - W_\infty(\mu_2, p_1)|^m + |W_\infty(\mu_2, p_1) - W_\infty(\mu_2, p_2)|^m]).
\]

It suffices to prove that there exist two constant $B_1, B_2 > 0$ such that for $1 \leq m \leq 4$,
\[ E[|W_\infty(\mu_1, p_1) - W_\infty(\mu_2, p_1)|^m] \leq B_1|\mu_1 - \mu_2|^m;
\]
\[ E[|W_\infty(\mu_2, p_1) - W_\infty(\mu_2, p_2)|^m] \leq B_2|p_1 - p_2|^m.
\]
Without loss of generality, assume $\mu_1 < \mu_2$. We now construct two stationary sequences \{(W_{n}^{\mu_i}: n \leq 0), i = 1, 2\} that are coupled "from the past". Let $V_j$ and $U_j$ be two i.i.d sequences corresponding to the service and inter-arrival times. For each $i$, we define a random walk:

$$Y_n^{\mu_i} = 0, \quad Y_n^{\mu_i} = \sum_{j=1}^{n} \left( \frac{V_j}{\mu_i} - \frac{U_j}{\lambda(p_1)} \right), \quad \forall n \geq 1.$$ 

It is clear that $Y_n^{\mu_i}$ is a random walk with negative drift for $i = 1, 2$. Define

$$W_{-n}^{\mu_i} = \max_{j \geq n} Y_j^{\mu_i} - Y_n^{\mu_i}, n \geq 0.$$ 

It is known in literature (see, for example, Blanchet and Chen (2015)) that $W_{-n}^{\mu_i}$ is a stationary waiting time process of a GI/GI/1 queue, starting from $-\infty$, with parameter $(\mu_i, p_1)$. In particular, the dynamics of $W_{-n}^{\mu_i}$ satisfies that

$$W_{-n+1}^{\mu_i} = \left( W_{-n} + \frac{V_n}{\mu_i} - \frac{U_n}{\lambda(p_1)} \right)^+, \quad \text{for } n \geq 1,$$

with $V_n/\mu_i$ being the service time of customer $-n$ and $U_n/\lambda(p_1)$ being the inter-arrival time between customer $-n$ and $-n+1$. For a fixed sequence of $(V_n, U_n)$, we have

$$W_0^{\mu_1} = \max_{j \geq 0} Y_j^{\mu_1}, \quad \text{and } W_0^{\mu_2} = \max_{j \geq 0} Y_j^{\mu_2}.$$ 

As $Y^{\mu_1} \geq Y^{\mu_2}$, we have $W_0^{\mu_1} \geq W_0^{\mu_2}$. Besides, let $\tau = \arg \max_{j \geq 0} Y_j^{\mu_1}$, we have

$$W_0^{\mu_1} - W_0^{\mu_2} = \max_{j \geq 0} Y_j^{\mu_1} - \max_{j \geq 0} Y_j^{\mu_2} = Y_\tau^{\mu_1} - \max_{j \geq 0} Y_j^{\mu_2} \leq Y_\tau^{\mu_1} - Y_\tau^{\mu_2}.$$ 

As a consequence, we have

$$|W_0^{\mu_1} - W_0^{\mu_2}| \leq \sum_{n=1}^{\tau} \left( \frac{V_n}{\mu_1} - \frac{V_n}{\mu_2} \right) \leq \frac{\mu_2 - \mu_1}{\mu_1} \sum_{n=1}^{\tau} V_n, \quad \text{with } \tau = \inf\{ n : W_{-n} = 0 \}.$$ 

Note that $V_n/\mu_1$ is the service time of customer $-n$ in the system with parameter $(p_1, \mu_1)$. By the definition of $\tau$, customer $-\tau$ enters service immediately upon the arrival and the queue remains busy by arrival of customer 0. Therefore, the summation of service times on the right hand side equals to the time between the arrival of customer $-\tau$ and the departure of customer $-1$, which equals to the observed busy period at the arrival of customer 0 plus its waiting time, i.e.

$$|W_0^{\mu_1} - W_0^{\mu_2}| \leq \frac{\mu_2 - \mu_1}{\mu_1} \sum_{n=1}^{\tau} V_n = \frac{\mu_2 - \mu_1}{\mu_1} (X_0^{\mu_1} + W_0^{\mu_1}).$$
Therefore, for each $n$,

$$
\mathbb{E}[|W_{0}^{\mu_{1}} - W_{0}^{\mu_{2}}|^{m}] \leq \frac{(\mu_{2} - \mu_{1})^{m}}{\mu_{1}^{m}} \mathbb{E}[(X_{0}^{\mu_{1}} + W_{0}^{\mu_{1}})^{m}] \leq \frac{(\mu_{2} - \mu_{1})^{m}}{\mu_{1}^{m}} \mathbb{E}[(X_{0}^{\mu_{1}} + W_{0}^{\mu_{1}})^{m}].
$$

Following Lemma 1, $\mathbb{E}[(X_{0}^{\mu_{1}} + W_{0}^{\mu_{1}})^{m}] \leq 2^{m}M$. Let $B_{1} = \max_{1 \leq m \leq 4} 2^{m}M/\mu_{1}^{m}$ and we conclude, for $1 \leq m \leq 4$,

$$
\mathbb{E}[|W_{0}^{\mu_{1}} - W_{0}^{\mu_{2}}|^{m}] \leq B_{1} |\mu_{1} - \mu_{2}|^{m}.
$$

The bound for $\mathbb{E}[(W_{\infty}(\mu_{2}, p_{1}) - W_{\infty}(\mu_{2}, p_{2}))^{m}]$ follows a similar argument and therefore we only provide a sketch of the proof. Without loss of generality, we assume $p_{1} < p_{2}$ and consider two stationary waiting time process $\{(W_{n}^{\mu_{i}} : n \leq 0), \lambda_{i} = \lambda(p_{i}), i = 1, 2\}$ that are coupled from past with the same sequence $(V_{n}, U_{n})$ in a similar way as we introduced previously. Then, we have $|W_{0}^{\mu_{1}} - W_{0}^{\mu_{2}}| \leq (\lambda_{1} - \lambda_{2})X_{1}^{\lambda_{1}}/\lambda_{2}$, and therefore,

$$
\mathbb{E}[|W_{0}^{\mu_{1}} - W_{0}^{\mu_{2}}|^{m}] \leq B_{2}|p_{1} - p_{2}|^{m}, \text{ with } B_{2} = \max_{1 \leq m \leq 4, 2 \leq p \leq \bar{p}} (M|\lambda'(p)|^{m}/\lambda(\bar{p})^{m}).
$$

As a consequence, we can take

$$
B = 8 \max_{1 \leq m \leq 4} (2^{m}M/\mu_{1}^{m}) \vee \max_{1 \leq m \leq 4, 2 \leq p \leq \bar{p}} (M|\lambda'(p)|^{m}/\lambda(\bar{p})^{m}). \quad (EC.3)
$$

**EC.1.5. Full Proof of Theorem 1**

Recall that, for fixed $k$, we have decomposed $R_{1,k} = h_{0}(I_{1} + I_{2} + I_{3}) + I_{4}$. So it suffices to show that there exists positive constants $C_{i}$ independent of $k$ such that $|I_{i}| \leq C_{i}k^{-\alpha}$, for $i = 1, 2, 3, 4$.

First, by Corollary 1, we have

$$
|I_{1}| \leq A\mathbb{E}[|W_{0}^{k} - \tilde{W}_{0}^{k}|^{2}]^{1/2} \leq A \left(2\mathbb{E}[(W_{D_{k-1}}^{k-1} - \tilde{W}_{D_{k-1}}^{k-1})^{2}] + 2\mathbb{E}[(\tilde{W}_{D_{k-1}}^{k-1} - \tilde{W}_{0}^{k})^{2}]\right)^{1/2}.
$$

By Lemma 2,

$$
\mathbb{E}[|W_{D_{k-1}}^{k-1} - \tilde{W}_{D_{k-1}}^{k-1}|^{2}] \leq e^{-\gamma_{D_{k-1}}}A\mathbb{E}[|W_{0}^{k-1} - \tilde{W}_{0}^{k-1}|^{2}]^{1/2} \leq \frac{4AM}{(k - 1)^{2}} \leq \frac{16AM}{k^{2}},
$$

given our choice of $D_{k-1}$ and the uniform bound $\mathbb{E}[|W_{n}^{k}|^{4}], \mathbb{E}[(W_{\infty}(\mu, p)|^{4}] \leq M$ by Lemma 1. On the other hand,

$$
\mathbb{E}[|W_{D_{k-1}}^{k-1} - \tilde{W}_{0}^{k}|^{2}] \leq B\mathbb{E}[|\mu_{k-1} - \mu_{k}|^{2}] \leq BK_{2}k^{-2\alpha}.
$$

Therefore, $E[I_{1}] \leq k^{-\alpha}C_{1}$ for any $C_{1} \geq A\sqrt{32AM + 2BK_{2}}.$
As to $|I_2|$, again, following Corollary 1,

$$|I_2| \leq A \mathbb{E}[(W_{Q_k}^k - \tilde{W}_{Q_k}^k)^2]^{1/2} \leq A \left(2 \mathbb{E}[(W_{Q_k}^k - \tilde{W}_{Q_k}^k)^2] + 2 \mathbb{E}[(\tilde{W}_{Q_k}^k)^2] \right)^{1/2}.$$  

By Theorem 1.1 in (Kella and Ramasubramanian (2012)) and our analysis for $I_1$,

$$\mathbb{E}[(W_{Q_k}^k - \tilde{W}_{Q_k}^k)^2] \leq \mathbb{E}[(W_0^k - \tilde{W}_0^k)^2] \leq (32AM + 2BK_2)k^{-2\alpha}.$$  

Besides, by Lemma 3,

$$\mathbb{E}[(\tilde{W}_{Q_k}^k)^2] \leq B \mathbb{E}[(p_{k-1} - p_k)^2] \leq BK_2k^{-2\alpha}.$$  

As a consequence, $\mathbb{E}[I_2] \leq C_2k^{-\alpha}$ for any $C_2 \geq A\sqrt{64AM + 6BK_2}$.

For $|I_3|$, again, by Lemma 3, we have

$$|I_3| \leq \mathbb{E}[Q_k^2]^{1/2} \mathbb{E} \left[ (\mathbb{E}[W_\infty(\mu_k, p_{k-1})] - \mathbb{E}[W_\infty(\mu_k, p_k)])^2 \right]^{1/2}$$

$$\leq \mathbb{E}[Q_k^2]^{1/2} \mathbb{E} \left[ B(p_{k-1} - p_k)^2 \right]^{1/2} \leq \sqrt{MK_2Bk^{-\alpha}}.$$  

Now we turn to $I_4$. Recall that

$$I_4 = (D_k - \lambda_k \mathbb{E}[T_k])(h_0 \mathbb{E}[W_\infty(\mu_k, p_k)] + \frac{h_0}{\mu_k} - p_k) + \mathbb{E}[Q_k(p_k - p_{k-1})].$$  

Note that by Lemma 1,

$$|h_0 \mathbb{E}[W_\infty(\mu_k, p_k)] + \frac{h_0}{\mu_k} - p_k| \leq h_0M + h_0\mu^{-1} + \bar{p},$$  

and following Condition 2,

$$|\mathbb{E}[Q_k(p_k - p_{k-1})]| \leq \mathbb{E}[Q_k^2]^{1/2} \mathbb{E}[(p_k - p_{k-1})^2]^{1/2} \leq \sqrt{MK_2k^{-\alpha}}.$$  

So it suffices to show that $|D_k - \lambda(p_k)\mathbb{E}[T_k]| = O(k^{-\alpha})$. Recall that $T_k$ is the time for the $D_k$-th customer to enter service. Let $F_n^k$ be the inter-service time between the $(n-1)$-th and the $n$-th customers in cycle $k$. Then, $T_k = \sum_{k=1}^n F_n^k$. By definition, we have

$$F_n^k = \begin{cases} \frac{U_n^k}{\lambda_{k-1}} + W_n^k - W_{n-1}^k & \text{for } 1 \leq n \leq Q_k \\ \frac{U_n^k}{\lambda_k} + W_n^k - W_{n-1}^k & \text{for } Q_k + 1 \leq n \leq D_k. \end{cases}$$  

Therefore,

$$T_k = \sum_{k=1}^{D_k} F_n^k = \frac{1}{\lambda_{k-1}} \sum_{n=1}^{Q_k} U_n^k + \frac{1}{\lambda_k} \sum_{n=Q_k+1}^{D_k} U_n^k + W_{D_k}^k - W_0^k$$

$$= \frac{1}{\lambda_k} \sum_{n=1}^{D_k} U_n^k + W_{D_k}^k - W_0^k + \left(1 - \frac{\lambda_{k-1}}{\lambda_k}\right) \sum_{k=1}^{Q_k} \frac{U_n^k}{\lambda_{k-1}}.$$
As a consequence,

\[ |D_k - \lambda_k \mathbb{E}[T_k]| \leq \lambda_k |\mathbb{E}[W^k_{D_k}] - \mathbb{E}[W^k_0]| + \mathbb{E}\left[ |\lambda_k - \lambda_{k-1}| \sum_{k=1}^{Q_k} \frac{U^k_n}{\lambda_{k-1}} \right]. \]

Following Lemma 2 and 3,

\[ |\mathbb{E}[W^k_{D_k}] - \mathbb{E}[W^k_0]| \leq \mathbb{E}|W^k_{D_k} - \bar{W}^k_{D_k}| + \mathbb{E}|W^{k-1}_{D_{k-1}} - \bar{W}^{k-1}_{D_{k-1}}| + \mathbb{E}|\bar{W}^k_{D_k} - \bar{W}^{k-1}_{D_{k-1}}| \]

\[ \leq k^{-\alpha} (2B \sqrt{K_2} + M(2M + 2)). \]

Besides,

\[ \mathbb{E}\left[ |\lambda_k - \lambda_{k-1}| \sum_{k=1}^{Q_k} \frac{U^k_n}{\lambda_{k-1}} \right]^2 = \mathbb{E}[||\lambda_k - \lambda_{k-1}||^2] \mathbb{E}\left[ \left( \sum_{k=1}^{Q_k} \frac{U^k_n}{\lambda_{k-1}} \right)^2 \right] \]

\[ \leq \mathbb{E}[||\lambda_k - \lambda_{k-1}||^2] \mathbb{E}\left[ (W^k_0 + \frac{U^k_n}{\lambda_{k-1}})^2 \right] \]

\[ \leq K_2 k^{-2\alpha} \cdot (2M + 2\mathbb{E}[U^2_1] \Delta^{-2}) \]

where the last equality follows form Condition 2 and Lemma 1, and the second last inequality follows form the fact that the first \( Q_k - 1 \) customers in cycle \( k \) are those who arrive during the waiting period of customer 0. In summary, we can conclude that \( R_{1,k} \leq Kk^{-\alpha} \) with

\[ K = h_0 \left( A \sqrt{256AM + 24BK_2} + \sqrt{MK_2 B} \right) + \sqrt{MK_2} \]

\[ + \left( 2B \sqrt{K_2} + M(2M + 2) + \sqrt{K_2(2M + 2\mathbb{E}[U^2_1] \Delta^{-2})} \right) (h_0 M + h_0 \mu^{-1} + \bar{p}). \]  

(EC.4)

**EC.1.6. Convergence Rate of Observed Busy Period**

As an analogue of Lemma 2, we prove a uniform convergence rate for the observed busy period \( X_n \), which will be used to bound \( B_k \) and \( V_k \) of the gradient estimator (13) that involves terms of \( X^k_n \).

**Lemma EC.1.** Let \( X^1_n \) and \( X^2_n \) be the observed busy period of the two queueing systems coupled as in Lemma 2, with \( X^1_0, X^2_0 \leq \max \bar{X}_0 \) and \( W^1_0, W^2_0 \leq \bar{W}_0 \).

1. \( |X^1_n - X^2_n| \leq 1_{\{\max(\Gamma_1, \Gamma_2) > n\}} (\sum_{k=1}^{n} \tau_k + X^1_0 + X^2_0). \)

2. There exists a constant \( K_4 > 0 \) such that \( |\mathbb{E}[X^1_n - X^2_n]|^m \leq K_4 e^{-0.5\gamma n} n^4 \) for all \( n \geq 1 \) and \( m \leq 2 \).
Proof of Lemma EC.1 1. Following the argument in Lemma 2, if \( W_1^1 \geq W_1^2 \), we will have \( W_1^1 = W_1^2 = 0 \) and hence \( X_1^1 = X_1^2 = 0 \). Since the two systems share the same sequence of arrivals and service times, \( X_n^1 = X_n^2 \) for all \( n \geq \Gamma_1 \). Therefore,

\[
|X_n^1 - X_n^2| \leq 1_{\{\max(\Gamma_1, \Gamma_2) > n\}} |X_n^1 - X_n^2| \leq 1_{\{\max(\Gamma_1, \Gamma_2) > n\}} \left( \sum_{k=1}^{n} \tau_k + X_0^1 + X_0^2 \right).
\]

The last inequality follows from \( 0 \leq X_i^1 \leq X_i^1 + \sum_{k=1}^{n} \tau_k \) for \( i = 1, 2 \).

2. Following 1 and part 2 of Lemma 2, for \( m = 1, 2 \),

\[
\mathbb{E}[|X_n^1 - X_n^2|^m] \leq \mathbb{E} \left[ 1_{\{\max(\Gamma_1, \Gamma_2) > n\}} \left( \sum_{k=1}^{n} \tau_k + X_0^1 + X_0^2 \right)^m \right] \\
\leq \mathbb{P}(\max(\Gamma_1, \Gamma_2) > n)^{1/2} \mathbb{E} \left[ \left( \sum_{k=1}^{n} \tau_k + X_0^1 + X_0^2 \right)^{2m} \right]^{1/2}
\]

where

\[
\mathbb{P}(\max(\Gamma_1, \Gamma_2) > n) \leq e^{-n\gamma} \mathbb{E}[2 + e^{\theta W_1^1} + e^{\theta W_0^2}] \leq e^{-n\gamma}(2 + 2M),
\]

and

\[
\mathbb{E} \left[ \left( \sum_{k=1}^{n} \tau_k + X_0^1 + X_0^2 \right)^{2m} \right] \leq 3^{2m-1} \left( n^{2m} \mathbb{E} \left[ \frac{U_1^{2m}}{\lambda(p)^{2m}} \right] + \mathbb{E}[(X_0^1)^{2m}] + \mathbb{E}[(X_0^2)^{2m}] \right).
\]

Therefore,

\[
\mathbb{E}[|X_n^1 - X_n^2|^m] \leq K_4 e^{-0.5m\gamma} n^4,
\]

with \( K_4 = 3^m (\max_{1 \leq m \leq 2} \mathbb{E}[U_1^{2m}]/\lambda(p)^{2m} + 2M)^{1/2} (2 + 2M)^{1/2} \).

\[ \square \]

EC.1.7. Proof of Theorem 2

The proof follows an induction-based approach similar to Broadie et al. (2011). For simplicity of notation, we write \( \Delta_k = (k - 1)^{-\beta} \). Since \( x_{k+1} = \pi_B(x_k - \eta_k H_k) \), we have

\[
\mathbb{E}[[x_{k+1} - x^*]|x_k] \leq \mathbb{E}[[x_k - \eta_k H_k - x^*]|x_k]
= \|x_k - x^*\|^2 - 2\eta_k \mathbb{E}[H_k | x_k] \cdot (x_k - x^*) + \eta_k^2 \mathbb{E}[\|H_k\|^2 | x_k]
= \|x_k - x^*\|^2 - \eta_k \nabla f(x_k) \cdot (x_k - x^*) - 2\eta_k \left( \mathbb{E}[H_k | x_k] - \frac{1}{2} \nabla f(x_k) \right) \cdot (x_k - x^*) + \eta_k^2 \mathbb{E}[\|H_k\|^2 | x_k]
\leq \|x_k - x^*\|^2 - \eta_k K_0 \|x_k - x^*\|^2 + 2\eta_k \mathbb{E}[\|H_k\| | x_k] - \frac{1}{2} \nabla f(x_k) \cdot (x_k - x^*) + \eta_k^2 \mathbb{E}[\|H_k\|^2 | x_k]
\leq \|x_k - x^*\|^2 - \eta_k K_0 \|x_k - x^*\|^2 + \eta_k (1 + \|x_k - x^*\|^2) B_k + \eta_k^2 \mathcal{V}_k,
\]

where \( K_0 = \max_{1 \leq m \leq 2} \mathbb{E}[U_1^m]/\lambda(p)^m \).
where the last inequality follows from $\|x_k - x^*\| \leq (1 + \|x_k - x^*\|^2)/2$ and the definition of $B_k$ and $V_k$. Let $b_k = \mathbb{E}[\|x_k - x^*\|^2]$ for all $k \geq 1$, then we have the following recursive inequality:

$$b_{k+1} \leq (1 - K_0\eta_k + \eta_k B_k)b_k + \eta_k B_k + \eta_k^2 V_k.$$  

By Condition 2 and 3, we have

$$b_{k+1} \leq (1 - K_0\eta_k + \eta_k B_k)b_k + \eta_k B_k + \eta_k^2 V_k \leq \left(1 - K_0\eta_k + \frac{K_0}{8}\eta_k\Delta_k\right)b_k + \frac{K_0}{8}\eta_k\Delta_k + K_3\eta_k\Delta_k.$$  

By Condition 1, $\Delta_{k-1}/\Delta_k \leq 1 + \frac{K_0}{2}\eta_k$, and by the induction assumption $b_k \leq C\Delta_{k-1}$, we have

$$b_{k+1} \leq \left(1 - K_0\eta_k + \frac{K_0}{8}\eta_k\Delta_k\right)\left(1 + \frac{K_0\eta_k}{2}\right)C\Delta_k + \frac{K_0}{8}\eta_k\Delta_k + K_3\eta_k\Delta_k$$

$$\leq C\Delta_k - \eta_k\Delta_k \left(\frac{K_0C}{2} - \frac{K_0C}{8}\Delta_k - \frac{K_0^2C}{16}\eta_k\Delta_k - \frac{K_0}{8} - K_3\right)$$

Then, we have $b_{k+1} \leq C\Delta_k$ as long as

$$\frac{K_0C}{2} - \frac{K_0C}{8}\Delta_k - \frac{K_0^2C}{16}\eta_k\Delta_k - \frac{K_0}{8} - K_3 \geq 0.$$  

(EC.5)

To check (EC.5), note that $\Delta_k, \eta_k K_0, K_0 \leq 1$ and $C \geq 8K_3/K_0$, therefore

$$\frac{K_0C}{2} - \frac{K_0C}{8}\Delta_k - \frac{K_0^2C}{16}\eta_k\Delta_k - \frac{K_0}{8} - K_3 \geq \frac{K_0C}{2} - \frac{K_0C}{8} - \frac{K_0C}{16} - \frac{K_0C}{8} - \frac{K_0C}{16} = \frac{K_0C}{16} > 0.$$

Then, we can conclude by induction,

$$\mathbb{E}[\|x_k - x^*\|^2] \leq C(k - 1)^{-\beta}.$$  

By Assumption 3, there exists $\theta_0 \in [0, 1]$ such that

$$|f(x_k) - f(x^*)| = |\nabla f(\theta_0(x_k - x^*) + x^*)^T(x_k - x^*)| \leq K_1\|x_k - x^*\|^2.$$  

As a consequence,

$$R_1(L) \leq \sum_{k=1}^{L} \mathbb{E}[T_k] K_1 Ck^{-\beta}.$$  

Note that $T_k$ equals to the arrival time of customer $D_k$ plus its waiting time. Therefore,

$$\mathbb{E}[T_k] \leq \frac{D_k}{\lambda_k} + \mathbb{E}[W_{D,k}] \leq \frac{D_k}{\lambda(\bar{p})} + M = O(D_k),$$

and we can conclude

$$R_1(L) = O\left(\sum_{k=1}^{L} D_k k^{-\beta}\right).$$

□
EC.1.8. Proof of Theorem 3

(i) According to Algorithm 1,

\[ h_k = \begin{cases} -\lambda(p_k) - p_k \lambda'(p_k) + h_0 \lambda'(p_k) \left[ \frac{1}{D_k(1 - \xi)} \sum_{n \geq \xi D_k} (X_n^k + W_n^k) + \frac{1}{\mu} \right], & Z = 1; \\ c'(\mu_k) - h_0 \frac{\lambda(p_k)}{\mu_k} \left[ \frac{1}{D_k(1 - \xi)} \sum_{n \geq \xi D_k} (X_n^k + W_n^k) + \frac{1}{\mu} \right], & Z = 2. \end{cases} \]

Note that \( \lambda(p) \), \( \lambda'(p) \) and \( \mu \) are bounded. Let \( C_0 = \max_{(\mu, \mu) \in B} \{ h_0 \lambda'(p_k), h_0 \lambda(p_k)/\mu \} \), then the bias

\[ B_k \leq C_0 \cdot \frac{1}{D_k(1 - \xi)} \sum_{n \geq \xi D_k} \left( E[X_n^k] - E[X_\infty(\mu_k, p_k)] + E[W_n^k] - E[W_\infty(\mu_k, p_k)] \right). \]

Following the two couplings constructed in Section 3, we can write

\[ E[W_n^k] - E[W_\infty(\mu_k, p_k)] = E[W_n^k] - E[\tilde{W}_n^k] + E[W_\infty(\mu, p_{k-1})] - E[W_\infty(\mu_k, p_k)], \]

for \( 1 \leq n \leq Q_k \), and \( E[W_n^k] - E[W_\infty(\mu_k, p_k)] = E[W_n^k] - E[\tilde{W}_n^k] \) for \( Q_k + 1 \leq n \leq D_k \).

In particular, for any \( n \geq \xi D_k \),

\[ |E[W_n^k] - E[W_\infty(\mu_k, p_k)]| \leq \mathbb{E}[|W_n^k - \tilde{W}_n^k| 1_{\{Q_k < 0.5 D_k\}}] + \mathbb{E}[|W_n^k - \tilde{W}_n^k| 1_{\{0.5 D_k \leq Q_k \leq \xi D_k\}}] + \mathbb{P}(Q_k \geq \xi D_k), \]

where, by Lemma 1 and Lemma 2,

\[ E\left[ |W_n^k - \tilde{W}_n^k| 1_{\{Q_k < 0.5 D_k\}} \right] \leq e^{-0.5 \gamma D_k} \mathbb{E}[\gamma Q_k (e^{\theta Q_k} + e^{\theta W_n^k}) |W_n^k - \tilde{W}_n^k| 1_{\{Q_k < 0.5 D_k\}}] \leq e^{-0.5 \gamma D_k} \mathbb{E}[\gamma Q_k (e^{\theta Q_k} + e^{\theta W_n^k}) |W_n^k - \tilde{W}_n^k|] \leq e^{-0.5 \gamma D_k} \mathbb{E}[\gamma^3 Q_k]^{1/3} (e^{\theta Q_k / 3} + e^{\theta W_n^k / 3})^{1/3} \mathbb{E}[|W_n^k - \tilde{W}_n^k|^{3}]^{1/3} \leq e^{-0.5 \gamma D_k} 4M. \]

Besides, by Theorem 1.1 of Kella and Ramasubramaniam (2012),

\[ \mathbb{E}[|W_n^k - \tilde{W}_n^k| 1_{\{0.5 D_k \leq Q_k \leq \xi D_k\}}] \leq \mathbb{E}[|W_n^k - \tilde{W}_n^k| 1_{\{0.5 D_k \leq Q_k \leq \xi D_k\}}] \leq \mathbb{E}[|W_n^k - \tilde{W}_n^k|^{2}]^{1/2} \mathbb{P}(Q_k \geq 0.5 \xi D_k)^{1/2}, \]

where, by Lemma 1,

\[ \mathbb{E}[|W_0^k - \tilde{W}_0^k|^{2}] \leq 2\mathbb{E}[|W_0^k|^{2}] + 2\mathbb{E}[\tilde{W}_0^2] \leq 4M, \]

\[ \mathbb{P}(Q_k \geq 0.5 \xi D_k) \leq e^{-0.5 \gamma D_k} \mathbb{E}[e^{\gamma Q_k}] \leq e^{-0.5 \gamma D_k} M. \]
Then, we have
\[ \mathbb{E}[|W_n^k - \tilde{W}_n^k| 1_{\{0.5\xi D_k \leq Q_k \leq \xi D_k\}}] \leq e^{-0.25\gamma \xi D_k} 2M. \]

Similarly,
\[ \mathbb{E}[|W_n^k - \tilde{W}_n^k| 1_{\{Q_k > \xi D_k\}}] \leq e^{-0.25\gamma \xi D_k} 2M. \]

In the end, by Lemma 1
\[ \|\mathbb{E}[W_\infty(\mu, p_{k-1})] - \mathbb{E}[W_\infty(\mu, p_k)]\| \leq \max(\mathbb{E}[W_\infty(\mu, p_{k-1})], \mathbb{E}[W_\infty(\mu, p_k)]) \leq M. \]

In summary, we have, for all \( n \geq \xi D_k \),
\[ |\mathbb{E}[W_n^k] - \mathbb{E}[W_\infty(\mu, p_k)]| \leq e^{-0.5\gamma \xi D_k} 4M + e^{-0.25\gamma \xi D_k} 4M + e^{-0.5\gamma \xi D_k} M \leq e^{-0.25\gamma \xi D_k} \cdot 9M. \]

For the observed busy period \( X_n^k \), following a similar analysis and Lemma EC.1, we have
\[
|\mathbb{E}[X_n^k] - \mathbb{E}[X_\infty(\mu, p_k)]| \\
\leq (2\mathbb{E}[(X_n^k)^2] + 2\mathbb{E}[(X_\infty(\mu, p_k))^2])^{1/2} \mathbb{P}(Q_k \geq 0.5\xi D_k)^{1/2} + K_4 e^{-0.5\gamma \xi D_k} D_k^4 \\
\leq e^{-0.25\gamma \xi D_k} (2M + K_4 D_k^4)
\]

Note that
\[ e^{-0.125\gamma \xi D_k} D_k^4 \leq \left( \frac{32}{\gamma \xi} \right)^4 e^{-4} \triangleq K_5. \]

If we choose
\[ D_k = \frac{\max(4(\log(144MC_0/K_0), 8\log((32M + 16K_4K_5)C_0/K_0)) + 8\log(k)})}{\gamma \xi}. \tag{EC.6} \]

Then,
\[ |\mathbb{E}[W_n^k] - \mathbb{E}[W_\infty(\mu, p_k)]| \leq \frac{K_0}{16C_0k}, \quad |\mathbb{E}[X_n^k] - \mathbb{E}[X_\infty(\mu, p_k)]| \leq \frac{K_0}{16C_0k}. \]

Therefore,
\[ B_k \leq \frac{C_0}{D_k(1 - \xi)} \sum_{n \geq \xi D_k} \left( |\mathbb{E}[X_n^k] - \mathbb{E}[X_\infty(\mu, p_k)]| + |\mathbb{E}[W_n^k] - \mathbb{E}[W_\infty(\mu, p_k)]| \right) \leq \frac{K_0}{8k}. \]

On the other hand, as \( \lambda(p) \), \( \lambda'(p) \) and \( \mu \) are bounded, \( C_1 \triangleq \max_{\mu, p \in B}\{|\lambda(p) + p\lambda'(p)|, |c'(\mu)|\} < \infty. \) Recall that \( C_0 = \max_{(\mu, p) \in B}\{h_0\lambda'(p_k), h_0\lambda(p)/\mu\}. \) Then,
\[ \mathbb{E}[|H_k|^2] \leq 2(C_1 + C_0/\mu)^2 + 2C_0^2 \mathbb{E} \left[ \frac{1}{(1 - \xi)^2 D_k^2} \left( \sum_{n \geq \xi D_k} (X_n^k + W_n^k) \right)^2 \right] \]
By Lemma 1, we have
\[
\mathbb{E} \left[ \frac{1}{(1 - \xi)^2 D_k^2} \left( \sum_{n \geq \xi D_k} (X_n^k + W_n^k) \right)^2 \right] \leq \mathbb{E} \left[ \frac{1}{(1 - \xi)^2 D_k^2} \left( \sum_{n \geq \xi D_k} \left( \frac{\lambda(p)}{\lambda(\bar{p})} \hat{X}_n^k + \hat{W}_n^k \right) \right)^2 \right],
\]
where \( \hat{W}_n^k \) and \( \hat{X}_n^k \) are defined as in Lemma 1. Note that by definition, \( \hat{W}_n^k \) and \( \hat{X}_n^k \) are stationary, we have
\[
\mathbb{E} \left[ \frac{1}{(1 - \xi)^2 D_k^2} \left( \sum_{n \geq \xi D_k} \left( \frac{\lambda(p)}{\lambda(\bar{p})} \hat{X}_n^k + \hat{W}_n^k \right) \right)^2 \right] \leq \frac{2}{(1 - \xi)^2 D_k^2} \mathbb{E} \left[ \left( \sum_{n \geq \xi D_k} \hat{X}_n^k \right)^2 \right] + \frac{2}{(1 - \xi)^2 D_k^2} \mathbb{E} \left[ \left( \sum_{n \geq \xi D_k} \hat{W}_n^k \right)^2 \right] \leq 2(1 - \xi)^{-2} \mathbb{E} \left[ \left( \frac{\lambda(p)}{\lambda(\bar{p})} \hat{X}_0^k \right)^2 \right] + 2(1 - \xi)^{-2} \mathbb{E}[(\hat{W}_0^k)^2] \leq 4(1 - \xi)^{-2} M.
\]
Therefore, \( V_k \) is uniformly bounded. Given that \( \eta_k = 2k^{-1}/K_0 \), we have \( \eta_k V_k \leq \frac{K_3}{k} \) with
\[
K_3 = (4(C_1 + C_0/\mu)^2 + 16C_0^2(1 - \xi)^{-2} M)/K_0. \tag{EC.7}
\]
(ii) Following the update rule, we immediately got
\[
\mathbb{E}[\|x_k - x_{k+1}\|^2] \leq \eta_k^2 \mathbb{E}[\|H_k\|^2] \leq k^{-2} K_3 / K_0.
\]
The last inequality follows that \( \mathbb{E}[\|H_k\|^2] \leq V_k \leq K_3 K_0. \)

(iii) We have just proved that the conditions of Theorem 1 are satisfied with \( \alpha = 1 \). Therefore, \( R_1(L) \leq K \sum_{k=1}^L k^{-1} \leq K \log(L) \) with the expression of \( K \) given in (EC.4). Besides, conditions of Theorem 2 are satisfied with \( \beta = 1 \) and \( D_k = O(\log(k)) \), therefore,
\[
R_2(L) \leq CK_1 \sum_{k=1}^L \left( \frac{D_k}{\lambda(\bar{p})} + M \right) k^{-1} = O(\log(L)^2).
\]
As a consequence, the total regret
\[
R(L) = R_1(L) + R_2(L) \leq K_{alg} \log(L)^2 \leq K_{alg} (M_L)^2, \text{ with } M_L = \sum_{k=1}^L D_k.
\]
In addition, we can find a closed-expression for \( K_{alg} \) as
\[
K_{alg} = K + C \frac{K_1}{\gamma \xi} \cdot (8 + \max(4(\log(112MC_0/K_0), 8\log((32M + 16K_4K_5)C_0/K_0) + M), \tag{EC.8}
\]
where \( K \) is defined by (EC.4) in which the term \( K_3 \) is defined by (EC.7), and \( C = \max\{\|x_0 - x^\ast\|^2, 8K_3/K_0\}. \)
EC.2. Additional Numerical Examples

In this section we conduct additional numerical experiments to confirm the practical effectiveness of our algorithm. In what follows, we first test the case where the uniform stability condition is relaxed; we next report the algorithm performance for GI/GI/1 queueing models with phase-type and lognormal distributions.

EC.2.1. Violation of Uniform Stability

We extend the M/M/1 example considered in Section 5.2 with the uniform stability condition relaxed. Specifically, we begin with an initial setting of \((p_0, \mu_0)\) such that \(\rho_0 \equiv \lambda(p_0)/\mu_0 = 2.55\), which violates the stability condition. As shown in Figure EC.1, the pricing and staffing policies \((p_k, \mu_k)\) remain convergent to \((p^*, \mu^*)\). Consistently, the resulting traffic intensity \(\rho_k \equiv \lambda(p_k)/\mu_k\) is quickly controlled below 1; that is, the workload is kept in check despite of the unstable performance in the initial cycle.
EC.2.2. M/G/1 with Phase-Type Service

To test the performance of our online learning algorithm for queues with non-exponential service times, we consider phase-type distributions: hyperexponential with $n$ phases ($H_n$) and Erlang with $n$ phases ($E_n$). Let $S$ be a generic service time, define the squared coefficient of variation (SCV) as the variance over the squared mean, i.e., $c_s^2 \equiv \text{Var}(S)/\mathbb{E}[S]^2$. In Figure EC.2 we report the convergent sequence $(p_k, \mu_k)$ with $H_2$ service with $c_s^2 = 8$ (top panel), $M$ service with $c_s^2 = 1$ (middle panel), and $E_8$ service with $c_s^2 = 1/8$ (bottom panel). Other parameters include the step length $\eta_k = 4/k$, cycle length $D_k = 20 + 10 \log(k)$ and initial condition $p_0 = 4$ and $\mu_0 = 12$ ($\lambda_0 = 5.249$).

Figure EC.2 confirms that our algorithm remains effective. In addition, the convergence is faster as the CSV $c_s^2$ decreases. This is intuitive because a less variable service-time distribution yields a smaller $\mathcal{V}_k$ for the gradient estimator.

EC.2.3. Lognormal Service and Arrival

Finally, we consider a $GI/GI/1$ queue with service and interarrival times following lognormal (LN) distributions, that is, an $LN/LN/1$ model. Our consideration here follows from the recent empirical confirmations of LN distributed service times in real service systems.

We let $c_s^2 = c_a^2 = 2$ with $c_a^2$ is the SCV of the LN-distributed interarrival times. The other parameters remain the same as in Section EC.2.2. Because the exact optimal solutions $(p^*, \mu^*)$ are unavailable for this model, we are unable to provide an estimate of the regret as done in Figure 4, nor can we confirm the convex structure of the problem. Nevertheless, Figure EC.3 shows that our online algorithm continues to work well, despite the fact that LN is no longer a light-tail distribution (Assumption 2 does not hold in this case).

EC.3. Additional Discussions on Assumption 3

In this section, we give more detailed discussions on Condition (a) of Assumption 3. We illustrate that such an assumption holds for a large class of $GI/GI/1$ models and general settings of cost function $c(\mu)$ and demand function $\lambda(p)$. Under some mild condition, the objective $f$ is smooth on $\mathcal{B} = [\underline{\mu}, \bar{\mu}] \times [\underline{p}, \bar{p}]$ and therefore, it remains to have

\[
\frac{\partial^2 f}{\partial \mu^2}(\mu, p) > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial p^2}(\mu, p) > 0 \quad \text{for all} \quad (\mu, p) \in \mathcal{B}.
\] (EC.9)
Joint pricing and staffing for an $M/G/1$ queue having (i) $H_2$ service with $c^2_2 = 8$ (top panel), (ii) $M$ service (middle panel), and (iii) Erlang service with $c_2^2 = 1/8$ (bottom panel). Other parameters are step length $\eta_k = 4/k$, cycle length $D_k = 20 + 10\log(k)$, initial condition $p_0 = 4$, $\mu_0 = 12$. The optimal pricing and staffing solutions are: (i) $(p^*, \mu^*) = (3.44, 16.86)$; (ii) $(p^*, \mu^*) = 3.40, 12.48$; (iii) $(p^*, \mu^*) = 3.38, 11.34$.

Then by the fact that $f$ is smooth and $B$ is compact, there must exists some $K_1 > 0$ such that

$$\frac{\partial^2 f}{\partial \mu^2}(\mu, p), \frac{\partial^2 f}{\partial p^2}(\mu, p) \geq K_1$$

for all $(\mu, p) \in B$. Hence, Condition (a) is guaranteed because,
given \( \nabla f(\mu^*, p^*) = 0 \), we have
\[
\frac{\partial f}{\partial \mu}(\mu, p)(\mu - \mu^*) \geq K_1(\mu - \mu^*)^2 \quad \text{and} \quad \frac{\partial f}{\partial p}(\mu, p)(p - p^*) \geq K_1(p - p^*)^2.
\]
Recall that
\[
f(\mu, p) = h_0 \mathbb{E}[Q_\infty(\mu, p)] + c(\mu) - p\lambda(p) \equiv h_0 q(\mu, \lambda(p)) + c(\mu) - p\lambda(p),
\]
where \( q(\mu, \lambda(p)) \equiv \mathbb{E}[Q_\infty(\mu, p)] \). Then we have
\[
\frac{\partial^2 f}{\partial \mu^2} = h_0 \frac{\partial^2 q}{\partial \mu^2} + c''(\mu),
\]
\[
\frac{\partial^2 f}{\partial p^2} = h_0 \frac{\partial^2 q}{\partial p^2} - 2\lambda'(p) - p\lambda''(p) = h_0 \frac{\partial^2 q}{\partial \lambda^2} \cdot (\lambda'(p))^2 + h_0 \frac{\partial q}{\partial \lambda} \cdot \lambda''(p) - 2\lambda'(p) - p\lambda''(p).
\]
We next give some simple sufficient conditions on the queueing model, cost function and demand function, which guarantee \( \frac{\partial^2 f}{\partial \mu^2} \) and \( \frac{\partial^2 f}{\partial \lambda^2} > 0 \).

Under Assumption 1, it can be proved that \( q(\mu, p) \) is convex in \( \mu \) (see, for example, Proposition 19 in L’Ecuyer and Glynn (1994)). Following a similar argument, we can also confirm that \( q(\mu, \lambda(p)) \) is also convex in \( \lambda \) (An alternative way to see that \( \frac{\partial^2 q}{\partial \mu^2} > 0 \) is via Lemma 4). In particular, by Little’s Law, we have \( q(\mu, p) = \lambda \mathbb{E}[W_\infty(\mu, \lambda(p))] \) and therefore, according to Lemma 4, \( \partial q/\partial \lambda = \mathbb{E}[W_\infty(\mu, \lambda)] + \mathbb{E}[X_\infty(\mu, \lambda)] \). Note that the steady-state mean waiting time and busy period are both strictly increasing in \( \lambda \), indicating that \( \partial q/\partial \lambda^2 > 0 \). A similar argument holds for \( \frac{\partial^2 q}{\partial \mu^2} > 0 \).

**Cost function** Given that \( \frac{\partial^2 q}{\partial \mu^2} > 0 \), as long as \( c(\mu) \) is convex, we can conclude \( \frac{\partial^2 f}{\partial \mu^2} > 0 \).

**Demand function** Now we provide some sufficient conditions that are easy to verify for some commonly used demand functions \( \lambda(p) \) such that \( \frac{\partial^2 f}{\partial p^2} > 0 \), given \( \frac{\partial^2 q}{\partial \lambda^2} > 0 \).

(i) If the demand function is linear, we have that \( \lambda'(p) \leq 0 \) and \( \lambda''(p) = 0 \), so that \( 2\lambda'(p) + p\lambda''(p) \leq 0 \). Therefore, \( \frac{\partial^2 f}{\partial p^2} > 0 \).

(ii) If the demand function is quadratic, i.e. \( \lambda(p) = ap^2 - 2bp + c \) for some \( a, b, c > 0 \) and \( p < b/a \). Then, we have \( \lambda''(p) > 0 \) and \( 2\lambda'(p) + p\lambda''(p) = 6ap - 4b \leq 0 \) if \( \bar{p} \leq 2b/3a \).

(iii) If the demand function is exponential, i.e. \( \lambda(p) = \exp(a - bx) \) for some \( a, b > 0 \), we have \( \lambda''(p) > 0 \) and \( 2\lambda'(p) + p\lambda''(p) = (-2 + pb)\lambda(p) \leq 0 \) as long as \( \bar{p}b \leq 2 \), i.e. \( \lambda(p)/\lambda(\bar{p}) \leq e^2 \approx 7.38 \). In other words, Assumption 3 holds if the demand will not change dramatically from the highest to the lowest price.