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ORIGINAL RESEARCH

A functional law of the iterated logarithm for multi-class queues with batch arrivals

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Abstract

A functional *law of the iterated logarithm* (LIL) and its corresponding LIL are established for a multiclass single-server queue with first come first served (FCFS) service discipline. The functional LIL and its LIL quantify the magnitude of asymptotic stochastic fluctuations of the stochastic processes compensated by their deterministic fluid limits. The functional LIL and LIL are established in three cases: underloaded, critically loaded and overloaded, for performance measures including the total workload, idle time, queue length, workload, busy time, departure and sojourn time processes. The proofs of the functional LIL and LIL are based on a strong approximation approach, which approximates discrete performance processes with reflected Brownian motions. Numerical examples are considered to provide insights on these limit results.

Keywords Functional law of the iterated logarithm · Law of the iterated logarithm · Multi-class queue · First come first served service discipline · Strong approximation

Mathematics Subject Classification 60K25 · 60F17 · 60B10 · 90B22

1 Introduction

In this paper, we develop a functional *law of the iterated logarithm* (LIL) and its corresponding LIL for a multi-class batch-arrival $(GI^B/GI)^K/1/\text{FCFS}$ queue, which has one server, K

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customer classes and first come first served (FCFS) service discipline, a renewal batch-arrival process (the first GI) and independent and identically distributed service times (the second GI).

In the literature of queueing theory, the multi-class batch-arrival queueing system under FCFS service discipline largely captured researchers' attention in recent 30 years because these queueing models are closely related to many actual applications in life, including data packetization in communication system, central processing unit (CPU) scheduling in computer system and packet-switched communication networks, see Kulkarni and Glazebrook (2002) and Whitt (1983a). Many stochastic approximation theories have been established for this type of queueing networks with arrival in batches and FCFS service discipline, including stationary probability distribution for single-server queue (Van Ommeren 1990) and infinite-server queue (Falin 1994; Whitt 1983b), and for multi-class discrete-time queueing system (Clercq et al. 2013), fluid approximation for multiclass queueing networks (Chen and Zhang 1997) and many-server queue (Whitt and Talreja 2008), diffusion approximation for multiclass queueing networks (Chen and Zhang 2000; Pang and Whitt 2012), strong approximation for feedforward queueing network (Chen and Shen 2000; Chen and Yao 2001), stationary optimal policies under a linear cost structure (Lee and Srinivasan 1989), and the auto-covariance analysis (Liu and Templeton 1993), etc. We mainly develop our results of functional LIL and its LIL with the help of strong approximations, so the most associated previous work to the current paper is Chen and Shen (2000), which obtained the strong approximation for multi-class $(GI/GI)^K/1/$FCFS model, and induce us to find the strong approximation for multi-class $(GI^B/GI)^K/1/$FCFS queueing system.

The LIL and functional LIL are belong to the field of stochastic process limit, which is the most commonly used method for the performance analysis of non-Markov queue. The $(GI^B/GI)^K/1/$FCFS queueing model considered here is a standard non-Markov queue. Hence, our work is related to the stochastic process limit. The first scientists to work in this direction include Kingman (1962), Prohorov (1963), Borovkov (1976, 1984), etc. These older generation of scientists provided the basic theoretical framework of performance analysis for the non-Markov queue by the stochastic process limits, which pointed out a scientific direction of queues for us until today. Base on these fundamental research frameworks and outcomes, later scientists did lots of excellent works in this topic. Iglehart and Whitt (1970a) studied the asymptotic performance of multiple channel queues in heavy traffic. Karpelevich and Kreinin (1981) investigated solution of diffusion equations for the two-phase queuing system $GI/G/1 \to G'/1/\infty$ in heavy traffic. Grigelionis and Mikulevičius (1987) investigated application of martingale methods in queueing system in heavy traffic, also see Pang et al. (2007). A separate direction was the recursive methods for the queueing LIL developed by Sakalauskas and Mikulevičius in multiphase systems (see Minkevičius and Steišūnas 2003) and open networks (see Minkevičius 2014; Sakalauskas and Minkevičius 2000) operating under different load conditions. This separate topic is very similar with our paper, the difference between them is the used method because we prove the LIL and functional LIL based on the strong approximation rather than the recursive method.

The functional LIL result was firstly developed by Strassen (1964) and established for the standard k-dimensional Brownian motion, denoted be W. Let $\{W_n(t), n = 3, 4, \ldots\}$ be a sequence of scaled k-dimensional Brownian motion with $W_n(t) = W(nt)/\sqrt{n \log\log n}$, Strassen showed that, with probability one, the sequence $\{W_n, n \geq 3\}$ is *relatively compact* (that is, every subsequence has its own convergent subsequence) in [0, 1] and that the limits of the convergent subsequences are contained in compact set, which is the set of absolutely continuous k-dimensional functions x satisfying $x(0) = 0$ and $\int_0^1 [\dot{x}(t)]^2 \, \mathrm{d}t \leq 1$, where the square is the inner product, $\dot{x}(t)$ denotes the derivative of x at t. In words, the compact set of all limit points is defined as

$$\left\{x \in \mathbb{C}^k[0,1] : x(0) = 0, \int_0^1 [\dot{x}(t)]^2 \, \mathrm{d}t \le 1\right\},$$

where $\mathbb{C}^k[0,1]$ denotes the functional space of continuous functions defined on $[0,1]$. Strassen's result of the functional LIL is the basis for this issue research, and has many applications in different research. Caramellino (1998) used Strassen's result to find the functional LIL of diffusion process. Tsai (2000) applied Strassen's result to Markov chain with a countable state space. For queues, Iglehart (1971) adopted Strassen's result to obtain the functional LIL for queue lengths, departures and waiting times of the multiple channel queueing systems in heavy traffic.

The LIL is an earlier result than the functional LIL and was developed by Lévy (1937, 1948). Compared with the functional LIL, the LIL give us more intuitively numerical representation. This earliest LIL is also for a standard Brownian motion W, that is, for a standard Brownian motion W,

$$\limsup_{t\to\infty} \frac{W(t)}{\sqrt{2t\log\log t}} = -\liminf_{t\to\infty} \frac{W(t)}{\sqrt{2t\log\log t}} = 1, \tag{1.1}$$

with probability one. The LIL in form (1.1) is called the strong version because the LIL in (1.1) provides an explicit value "1" on the right-hand side to quantify the asymptotic rate of the increasing variability for a standard Brownian motion. Based on the LIL for Brownian motion, various LIL results have later been developed for performance functions in queueing systems, such as multiple channel queues in heavy traffic (Iglehart 1971) by a method based on renewal process, the strictly overloaded tandem queueing model (assuming all queues are strictly overloaded) (Minkevičius and Steišūnas 2003) using a recursive method, the strictly overloaded generalized Jackson network (Minkevičius 2014; Sakalauskas and Minkevičius 2000), and queues with customer priorities (Guo and Liu 2015; Guo et al. 2018) based on strong approximation.

Supplement to the strong version of LIL (1.1) above, a weak version of LIL received researchers' wide attention in the literature. In contrast to the strong form in (1.1), Chen and Yao (2001) provided a weak form of LIL for the queue length process Q (centered by its fluid function $\bar{Q}$) of the $GI/GI/1$ queue: they showed that $\sup_{0\le t\le T}\left|Q(t)-\bar{Q}(t)\right|$ is in the same order of the function $\sqrt{T\log\log T}$ as $T\to\infty$. That is, $\sup_{0\le t\le T}\left|Q(t)-\bar{Q}(t)\right| = O(\sqrt{T\log\log T})$ with probability one. The result is called the weak form because the LIL limit [as in (1.1)] was not clearly identified. Beyond an independent theoretical result, the weak LIL is usually used as a tool to prove the strong approximation for queueing networks. See Chen and Mandelbaum (1994), Chen and Shen (2000) and Chen and Yao (2001) for more results on weak LILs.

Our contributions We summarize our contributions in three directions.

- First, we establish a functional LIL in Strassen's version and a LIL in Lévy's version as in (1.1) for the key system functions of the $(GI^B/GI)^K/1$/FCFS queue, including the total workload, idle time, queue length, workload, busy time, departure and sojourn time processes (see Sect. 2 for their definitions).
- Second, we cover all three cases of the traffic intensity ρ: (i) underloaded with $\rho < 1$, (ii) critically loaded with $\rho = 1$ and (iii) overloaded with $\rho > 1$. In terms of the model input parameters, we identify the LIL and the functional LIL of the above performance

measures as simple absolutely-continuous function sets and analytic functions, respectively.

- Third, our results on the functional LIL and the LIL limits provide interesting and sometimes counterintuitive observations. For example, Theorem 3.2 shows us that the Little's law between the queue length and the workload processes in the functional LIL version (the function-set-version) generally holds in the under loaded and overloaded cases and fails in the critically loaded case, and Theorem 3.3 presents a similar insight on the Little's law in the LIL version (the functional version)

The strong approximation approach We obtain the functional LIL and the LIL results based on the strong approximations. We first relate the functional LILs of the performance functions to the functional LILs of their strong approximations; we next is to obtain the desired functional LIL limits by analyzing the (reflected) Brownian motions given by the strong approximations. We briefly demonstrate the idea of strong approximation using a renewal process $\{N(t), t \geq 0\}$ with rate $\lambda > 0$ and interrenewal-time variance $\sigma^2 < \infty$. Let $\bar{N}(t) \equiv \lambda t$ and $\widetilde{N}$ be the SA for N. We write

$$N(t) \approx \widetilde{N}(t) \equiv \bar{N}(t) + \lambda^{3/2}\sigma W(t), \tag{1.2}$$

for a large $t > 0$, where $W(t)$ is a standard Brownian motion. In addition, the error of the strong approximation $N(t) - \widetilde{N}(t)$ is a higher order infinitesimal of $t^{1/r}$ with some $r > 2$, see Horváth (1984a, b). Strong approximations have been developed for various stochastic processes, such as random walks (Csörgő et al. 1987) and renewal-related processes (Csörgő and Révész 1981; Csörgő and Horváth 1993). There is a large volume of literature using the strong approximation to study queueing models, including the $GI/GI/1$ queue (Chen and Yao 2001), $GI/GI/\infty$ queue (Glynn and Whitt 1991a), multiple channel queue (Zhang et al. 1990), tandem-queue network (Glynn and Whitt 1991b), generalized Jackson network (Chen and Mandelbaum 1994; Horváth 1992; Zhang 1997), non-preemptive priority queue (Zhang and Hsu 1992), time-dependent Markovian network queues (Mandelbaum and Massey 1995; Mandelbaum et al. 1998) and feedforward queueing networks (Chen and Shen 2000; Chen and Yao 2001).

Organization of the paper In Sect. 2, we formally introduce the $(GI^B/GI)^K/1$/FCFS queue, and review its fluid limit, which are used to construct the FLILs and LILs. In Sect. 3, we present our main results Theorems 3.2 and 3.3 . We also provide insights of these results. In Sect. 4, we give the proofs of the main results and other supporting results. In Sect. 5, we provide concrete numerical examples to give some insights from an engineering perspective. Finally, we draw conclusions in Sect. 6.

Notations We now summarize all notations used below. All random variables and processes are defined on a common probability space $(\Omega, \mathcal{F}, \mathsf{P})$. We reserve $\mathsf{E}(\cdot)$ for expectation and $Var(\cdot)$ for variance. We write $X =_d Y$ if X and Y have the same distribution. We use $\mathbb{R}^k$ and $\mathbb{R}^k_+$ to denote the k-dimensional spaces of real and nonnegative real numbers. All vectors in $\mathbb{R}^k$ are understood to be column vectors, and the transpose of a vector or a matrix is denoted by a prime. Let $a \vee b \equiv \max\{a, b\}$, $a^+ \equiv \max\{a, 0\}$ and $\lfloor a \rfloor$ be the largest integer less than or equal to a. Let $\mathbb{D}^k[a, b]$ be the k-dimensional space of right-continuous functions on $[a, b]$ having left limits, endowed the Skorohod topology (e.g., see Ethier and Kurtz 1986). Let $\mathbb{C}^k[a, b]$ be the k-dimensional space of continuous functions. We say that $f_n \rightrightarrows \mathcal{K}_f$ *with probability 1* (w.p.1.) if the sequence of $\{f_n, n \geq 1\}$ is relatively compact (i.e., every subsequence has a convergent subsubsequence) and the set of all limit points is the compact set $\mathcal{K}_f$. Let $\|f\|_T \equiv \sup_{0 \leq t \leq T} |f(t)|$ be the uniform norm of f. We say $f_n \to f$ *uniformly on compact sets* (u.o.c.) if $\|f_n - f\|_T \to \infty$, as $n \to 0$. We say $f(t) = O(g(t))$

as $t \to \infty$ if $\limsup_{t\to\infty} |f(t)/g(t)| \le M$ for some $M > 0$ and $f(t) = o(g(t))$ as $t \to \infty$ if $\lim_{t\to\infty} |f(t)/g(t)| = 0$. We let $\varphi(t) = \sqrt{2t \log\log t}$ for all $t > e$ and $\eta(t) = 0$ for all $t \ge 0$.

2 The $(GI^B/GI)^K/1$ queueing model

The model $(GI^B/GI)^K/1$ is a single-server station queue serving K classes of jobs, $K \ge 1$. Customers of all classes arrive exogenously in batches, wait for service and leave the system after service completion. A FCFS service discipline is enforced among K classes, that is, customers in different batches are served in the order of arrival, and customers in the same batch are numbered by $1, 2, \ldots,$ and served in the order of the numbering. We assume the FCFS service discipline is work-conserving, i.e., the server can not stay idle if there is any customer waiting for service.

Primitive data For each class k, consider three independent and identically distributed sequences of nonnegative random variables $u = \{u_k(n), n = 1, 2, \ldots\}$, $\xi = \{\xi_k(j), j = 1, 2, \ldots\}$ and $v = \{v_k(i), i = 1, 2, \ldots\}$, where $u_k(n)$ is the interarrival time between the $(n-1)$th and nth batch arrivals, $\xi_k(j)$ is the jth batch size (number of customers in the jth batch), and $v_k(i)$ is the service time of the ith customer. Suppose that u, ξ and v are mutually independent. Let the means $\mathsf{E}[u_k(1)] \equiv 1/\lambda_k$, $\mathsf{E}[v_k(1)] \equiv 1/\mu_k$ and $\mathsf{E}[\xi_k(1)] \equiv m_k$, variances $Var[u_k(1)]$,$Var[v_k(1)]$ and $Var[\xi_k(1)]$, and squared coefficients of variation $c^2_{a,k} \equiv Var[u_k(1)]/(\mathsf{E}[u_k(1)])^2$, $c^2_{s,k} \equiv Var[v_k(1)]/(\mathsf{E}[v_k(1)])^2$ and $c^2_{b,k} \equiv Var[\xi_k(1)]/(\mathsf{E}[\xi_k(1)])^2$, respectively. Define the following partial sums

$$U_k(n) \equiv \sum_{i=1}^{n} u_k(i), \quad B_k(n) \equiv \sum_{i=1}^{n} \xi_k(i) \quad \text{and} \quad V_k(n) \equiv \sum_{i=1}^{n} v_k(i), \quad n = 1, 2, \ldots, \tag{2.1}$$

where $U_k(n)$ is the arrival time of the nth class-k batch, $V_k(n)$ is the total service time of the first n class-k customers and $B_k(n)$ is the total number of class-k customers of the first n batches. Define the two renewal counting processes

$$A_k(t) \equiv \max\{n \ge 0 : U_k(n) \le t\} \quad \text{and} \quad S_k(t) \equiv \max\{n \ge 0 : V_k(n) \le t\}, \tag{2.2}$$

where $A_k(t)$ counts the total number of batches arriving in $[0, t]$ and $S_k(t)$ denotes the number of class-k service completions in $[0, t]$ if the server always serves the kth class. Therefore, $B_k(A_k(t))$ is the total number of class-k customers arriving in $[0, t]$.

Define the traffic intensity

$$\rho \equiv \sum_{k=1}^{K} \rho_k \quad \text{where} \quad \rho_k \equiv \frac{m_k \lambda_k}{\mu_k}, \quad k = 1, 2, \ldots, K. \tag{2.3}$$

We say the system is underloaded when $\rho < 1$, critically loaded when $\rho = 1$ and overloaded when $\rho > 1$.

Performance measures For $k = 1, 2, \ldots, K$, at time t, let queue length $Q_k(t)$ be the number of class k customers in system, $W_k(t)$ be the workload (that is the amount of unfinished work) of class k customers, $Z(t) = \sum_{k=1}^{K} W_k(t)$ be the total workload, $T_k(t)$ be the total amount of time that the server serves class-k customers in $[0, t]$, $Y(t) = t - \sum_{k=1}^{K} T_k(t)$ be the server's total idle time in $[0, t]$, $D_k(t) = S_k(T_k(t))$ be the number of class-k departures in $[0, t]$. Define the potential sojourn time $\mathcal{T}_{k,i}(t)$ as the sojourn time (total time in the system) of the

ith customer in a class-k batch which arrives at t, $i = 1, 2, \ldots$. These performance measures satisfy the following equations

$$Q_k(t) = B_k(A_k(t)) - D_k(t) \geq 0, \tag{2.4}$$

$$W_k(t) = V_k(B_k(A_k(t))) - V_k(D_k(t)) - \nu_k(t) \geq 0, \tag{2.5}$$

$$\int_0^\infty Z(t)\mathrm{d}Y(t) = 0, \tag{2.6}$$

$$\mathcal{T}_{k,i}(t) = Z(t) + \sum_{j=1}^{i} v_k^*(j), \quad k = 1, 2, \ldots, K, \quad i = 1, 2, \ldots, \tag{2.7}$$

where $\nu_k(t)$ is the elapsed service time of the customer in service if the server is serving a class-k customer at t and is 0 otherwise, and $\{v_k^*(j), j \geq 1\}$ is a copy of $v_k(1)$ and is assumed to be independent and identically distributed.

Remark 2.1 [Explanation for (2.4)–(2.7)] Equations (2.4) and (2.5) hold by flow conservation. Equation (2.6) holds because the service discipline is work conserving, namely, when the server is idle at time t (i.e., $Y(t)$ increases) only when the system is empty (i.e., $Z(t) = 0$). We note that an equivalent representation for $Y(t)$ and $Z(t)$ is

$$Z(t) = \Phi(N)(t), \quad Y(t) = \Psi(N)(t), \quad \text{and} \quad N(t) \equiv \sum_{k=1}^{K} V_k(B_k(A_k(t))) - t, \tag{2.8}$$

where the two functions (Ψ, Φ) defined by

$$\Psi(x)(t) = \sup_{0 \leq s \leq t} [-x(s)]^+ \quad \text{and} \quad \Phi(x)(t) = x(t) + \Psi(x)(t),$$

are known as the one dimensional oblique reflection mapping [see Harrison (1985) for discussions on the oblique reflection mapping]. Finally, Eq. (2.7) means that if a class-k batch arrives at time t, the sojourn time of the ith customer in that batch is the workload $Z(t)$ plus the total service times of the first i customers in that batch.

Fluid limits of performance measures In order to define the LIL- and functional LIL-scaling, we next introduce the the fluid limits of the performance measures, which is from the functional strong law of large numbers for renewal process. In order to find the fluid limits, we need other performance measure: the arrival time of the last customer who has finished service by t, denoted by $\tau(t)$, whose fluid scaling process is defined to be $\bar{\tau}^{(n)}(t) \equiv \tau(nt)/n$. Define other fluid-scaled processes as

$$\bar{Z}^{(n)}(t) \equiv \frac{1}{n} Z(nt), \quad \bar{Y}^{(n)}(t) \equiv \frac{1}{n} Y(nt), \quad \bar{N}^{(n)}(t) \equiv \frac{1}{n} N(nt), \quad \bar{\mathcal{T}}_{k,i}^{(n)}(t) \equiv \frac{1}{n} \mathcal{T}_{k,i}(nt),$$

$$\bar{Q}_k^{(n)}(t) \equiv \frac{1}{n} Q_k(nt), \quad \bar{W}_k^{(n)}(t) \equiv \frac{1}{n} W_k(nt), \quad \bar{T}_k^{(n)}(t) \equiv \frac{1}{n} T_k(nt), \quad \bar{D}_k^{(n)}(t) \equiv \frac{1}{n} D_k(nt),$$

for all $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots$. We next summarize the fluid limits of the system $(GI^B/GI)^K/1$, see Chen and Zhang (2000) and Dai (1995) for proof and more general cases.

Lemma 2.1 [Fluid limits for the $(GI^B/GI/1)^K$ queue (Chen and Shen 2000)] *Assume* $\mathsf{E}[u_k(1)] < \infty$ *and* $\mathsf{E}[v_k(1)] < \infty$. *If the system is initially empty, then for all* $k = 1, \ldots, K$, $i = 1, 2, \ldots$,

$$\left(\bar{Z}^{(n)}, \bar{Y}^{(n)}, \bar{N}^{(n)}, \bar{\tau}^{(n)}, \bar{Q}_k^{(n)}, \bar{W}_k^{(n)}, \bar{D}_k^{(n)}, \bar{T}_k^{(n)}, \bar{\mathcal{T}}_{k,i}^{(n)}\right)$$

$$\to \big(\bar{Z}, \bar{Y}, \bar{N}, \bar{\tau}, \bar{Q}_k, \bar{W}_k, \bar{D}_k, \bar{T}_k, \bar{\mathcal{T}}_{k,i}\big) \equiv \bar{\mathbb{X}}_k, \quad \textit{u.o.c., w.p.1}, \quad \textit{as } n \to \infty,$$

where $\bar{\mathbb{X}}_k(t)$ *satisfies*

$$\bar{T}_k(t) \equiv \frac{m_k\lambda_k}{\mu_k(1 \vee \rho)}t, \quad \bar{Q}_k(t) \equiv \mu_k \bar{W}_k(t) \equiv \left(1 - \frac{1}{\rho}\right)^+ m_k\lambda_k t, \quad \bar{D}_k(t) \equiv m_k\lambda_k\bar{\tau}(t) \equiv \frac{m_k\lambda_k}{1 \vee \rho}t,$$
$$\bar{Z}(t) \equiv \bar{\mathcal{T}}_{k,i}(t) \equiv (\rho - 1)^+ t, \quad \bar{Y}(t) \equiv (1-\rho)^+ t, \quad \bar{N}(t) \equiv (\rho - 1)t, \quad \bar{\tau}(t) \equiv \frac{t}{1 \vee \rho}. \tag{2.9}$$

The objective of the rest of the paper is to establish the LIL and functional LIL for performance functions $\big(Z, Y, Q_k, W_k, T_k, D_k, \mathcal{T}_{k,i}, 1 \le k \le K, i = 1, 2, \dots\big)$ and identify the LIL and functional LIL limits as functions of the model input data

$$\mathcal{D} \equiv \big(\lambda_k, m_k, \mu_k, c_{a,k}^2, c_{b,k}^2, c_{s,k}^2, 1 \le k \le K\big). \tag{2.10}$$

3 Main results

We now establish the LIL and the functional LIL for the $(GI^B/GI)^K/1$ queue. We first define the LIL and functional LIL scalings in Sect. 3.1; next in Sect. 3.2 we develop the results of the LIL and functional LIL in the underloaded, critically loaded and overloaded cases. All proofs are postponed in Sect. 4. For the convenience of readers, we gives several remarks immediately following main results.

3.1 The LIL and functioinal LIL scaling

LIL limits We define

$$Z^*_{\sup} = \limsup_{t\to\infty} \frac{Z(t) - \bar{Z}(t)}{\varphi(t)} \quad \text{and} \quad Z^*_{\inf} = \liminf_{t\to\infty} \frac{Z(t) - \bar{Z}(t)}{\varphi(t)}. \tag{3.1}$$

Similar to (3.1), for $k = 1, 2, \dots, K$ and $i = 1, 2, \dots,$ we define the following upper and lower LIL limits:

$$Y^*_{\sup}, Q^*_{k,\sup}, W^*_{k,\sup}, T^*_{k,\sup}, D^*_{k,\sup}, \mathcal{T}^*_{k,i,\sup}, Y^*_{\inf}, Q^*_{k,\inf}, W^*_{k,\inf}, T^*_{k,\inf}, D^*_{k,\inf}, \mathcal{T}^*_{k,i,\inf}. \tag{3.2}$$

We try to express all the LIL limits (3.2) in terms of the input data $\mathcal{D}$ in (2.10).
Functional LIL limits For $k = 1, 2, \dots, K$ and $i = 1, 2, \dots,$ let

$$Z^n(t) = \frac{Z(nt) - \bar{Z}(nt)}{\varphi(n)} \quad \text{and} \quad \mathcal{T}^n_{k,i}(t) = \frac{\mathcal{T}_{k,i}(nt) - \bar{\mathcal{T}}_{k,i}(nt)}{\varphi(n)}. \tag{3.3}$$

We define the functional LIL-scaled processes $Q^n_k(t)$, $X^n(t)$, $D^n_k(t)$, $W^n_k(t)$, $T^n_k(t)$ and $Y^n(t)$ in the same token of (3.3). We try to develop the functional LIL results in the following form: for $k = 1, 2, \dots, K$ and $i = 1, 2, \dots,$ as $n \to \infty$,

$$\big(Z^n, Y^n, Q^n_k, W^n_k, T^n_k, D^n_k, \mathcal{T}^n_{k,i}\big) \rightrightarrows \big(\mathcal{K}_Z, \mathcal{K}_Y, \mathcal{K}_{Q_k}, \mathcal{K}_{W_k}, \mathcal{K}_{T_k}, \mathcal{K}_{D_k}, \mathcal{K}_{\mathcal{T}_{k,i}}\big) \equiv \mathcal{K}^*_k, \tag{3.4}$$

w.p.1., where all sets $\mathcal{K}_Z, \mathcal{K}_Y, \mathcal{K}_{Q_k}, \mathcal{K}_{W_k}, \mathcal{K}_{T_k}, \mathcal{K}_{D_k}, \mathcal{K}_{\mathcal{T}_{k,i}}$ are compact sets expressed by the data $\mathcal{D}$ in (2.10) and some compact set $\mathcal{G}$ defined as follows.

Spaces of functional LIL limits For $\delta > 0$, let $\mathcal{G}_k(\delta)$ be the space of absolutely continuous functions having quadratic variations no larger than δ^2, in particular,

$$\mathcal{G}_k(\delta) \equiv \left\{ x \in \mathbb{C}^k[0,1] : x(0) = 0, \int_0^1 [\dot{x}(t)]^2 \, \mathsf{d}t \le \delta^2 \right\}, \tag{3.5}$$

where the square denotes inner product, and $\dot{x}(t)$ denotes the derivative of $x(t)$ which exists almost everywhere with respect to Lebesgue measure. We let $\mathcal{G} \equiv \mathcal{G}_1$.

For instance, $x_1(t) = \delta a t$ belongs to $\mathcal{G}(\delta)$ for $a \le 1$ because $x_1(0) = 0$ and $\int_0^1 [\dot{x}_1(t)]^2 \mathsf{d}t = a^2\delta^2 \le \delta^2$; $x_2(t) \equiv (\delta b_1 t, \delta b_2 t)'$ belongs to $\mathcal{G}_2(\delta)$ for $b_1^2 + b_2^2 \le 1$, because $x_2(0) = 0$ and $\int_0^1 [\dot{x}_2(t)]^2 \mathsf{d}t = \delta^2(b_1^2 + b_2^2) \le \delta^2$.

The set $\mathcal{G}_k(\delta)$ defined above is a compact set in $\mathbb{C}^k[0,1]$: for $x \in \mathcal{G}_k(\delta)$ and $0 \le c \le d \le 1$,

$$|x(d) - x(c)| = \left| \int_c^d \dot{x}(t) \mathsf{d}t \right| \le \left(\int_c^d [\dot{x}(t)]^2 \mathsf{d}t \right)^{1/2} \le \delta (d-c)^{1/2}, \tag{3.6}$$

where $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^k$.

Remark 3.1 (Scaling of LIL and the functional LIL) The LIL limits provide asymptotic upper and lower bounds because they are established by letting the time $t \to \infty$ [as in (3.1)]. As opposed to the constant LIL limit, the functional LIL limits are obtained as compact functional spaces containing the limits of all convergent subsequences, as the scaling $n \to \infty$. The functional LIL provides asymptotic bounds for all possible functional limits. In this sense, the LIL limits can be considered special cases of the functional LIL limits.

3.2 The LIL and functional LIL Limits

Throughout the rest of the paper, suppose that, for all $k = 1, 2, \ldots, K$, for some $r > 2$,

$$\mathsf{E}\left[u_k(1)^r\right] < \infty, \quad \mathsf{E}\left[v_k(1)^r\right] < \infty \quad \text{and} \quad \mathsf{E}\left[\xi_k(1)^r\right] < \infty. \tag{3.7}$$

We first present the LIL and the functional LIL of the batch arrival process.

Theorem 3.1 (The LIL and functional LIL for compound renewal arrival processes) *Suppose* (3.7) *holds. For A_k, B_k defined above, we have, w.p.1,*

$$\begin{aligned} &\limsup_{t\to\infty} \frac{B_k(A_k(t)) - \lambda_k m_k}{\varphi(t)} = -\liminf_{t\to\infty} \frac{B_k(A_k(t)) - \lambda_k m_k}{\varphi(t)} = m_k \sqrt{\lambda_k \left(c_{a,k}^2 + c_{b,k}^2\right)}, \\ &\frac{B_k(A_k(nt)) - \lambda_k m_k}{\varphi(n)} \rightrightarrows m_k \sqrt{\lambda_k \left(c_{a,k}^2 + c_{b,k}^2\right)}\, \mathcal{G}(1), \quad \textit{as} \quad n \to \infty. \end{aligned} \tag{3.8}$$

Next, we give the functional LIL results for performance measures Z, Y, Q_k, W_k, T_k, D_k, $\mathcal{T}_{k,i}$, $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots$ in three cases: underloaded, critically loaded and overloaded. Let $c_k^2 \equiv m_k(c_{a,k}^2 + c_{b,k}^2) + c_{s,k}^2$ be the variability coefficient for class k (capturing the variabilities of the arrival, batch and service distributions). Let $\hat{\rho}_K \equiv \sum_{k=1}^K \rho_k c_k^2/\mu_k$,

$$d_k^{*2} \equiv \left(1 - \frac{\rho_k}{\rho}\right)^2 \frac{m_k^2 \lambda_k \left(c_{a,k}^2 + c_{b,k}^2\right)}{\rho} + \frac{\rho_k^3 \mu_k c_{s,k}^2}{\rho^3} + \frac{m_k^2 \lambda_k^2}{\rho^3} \sum_{i \ne k} \frac{\rho_i c_i^2}{\mu_i}, \tag{3.9}$$

$$q_k^{*2} \equiv \left[1 - \frac{1}{\sqrt{\rho}}\left(1 - \frac{\rho_k}{\rho}\right)\right]^2 \lambda_k m_k^2 \left(c_{a,k}^2 + c_{b,k}^2\right) + \frac{\rho_k^3}{\rho^3}\mu_k c_{s,k}^2 + \frac{\lambda_k^2 m_k^2}{\rho^3}\sum_{i \neq k}\frac{\rho_i c_i^2}{\mu_i}, \tag{3.10}$$

$$w_k^{*2} \equiv \left[1 - \frac{1}{\sqrt{\rho}}\left(1 - \frac{\rho_k}{\rho}\right)\right]^2 \frac{\rho_k c_k^2}{\mu_k} + \frac{\rho_k^2}{\rho^3}\sum_{i \neq k}\frac{\rho_i c_i^2}{\mu_i}, \tag{3.11}$$

$$t_k^{*2} \equiv \left(1 - \frac{\rho_k}{\rho}\right)^2 \frac{\rho_k c_k^2}{\rho \mu_k} + \frac{\rho_k^2}{\rho^3}\sum_{i \neq k}\frac{\rho_i c_i^2}{\mu_i}. \tag{3.12}$$

Theorem 3.2 (The functional LIL for the $(GI^B/GI)^K/1$ queue) *Suppose* (3.7) *holds.* (i) *If* $\rho < 1$, *then the functional LIL in* (3.4) *holds with*

$$\mathcal{K}_k^* = \left\{\left(\eta, \sqrt{\hat{\rho}_K}x, \eta, \eta, \sqrt{\frac{\rho_k c_k^2}{\mu_k}}x, m_k \lambda_k^{1/2}\sqrt{c_{a,k}^2 + c_{b,k}^2}x, \eta\right) : x \in \mathcal{G}(1)\right\}, \tag{3.13}$$

where η *is a zero operator, that is,* $\eta(x) = 0$ *for all* $x \in \mathcal{G}(\delta)$ *with* $\delta > 0$.
(ii) *If* $\rho = 1$, *the functional LIL in* (3.4) *holds with*

$$\begin{aligned}
&(\mathcal{K}_Z, \mathcal{K}_Y, \mathcal{K}_{Q_k}, \mathcal{K}_{W_k}, \mathcal{K}_{D_k}, \mathcal{K}_{\mathcal{T}_{k,i}}) \\
&\quad = \left\{\left(\Phi(x), \Psi(x), \lambda_k m_k \Phi(x), \rho_k \Phi(x), G_k^d(y,z), \Phi(x)\right) : x \in \mathcal{G}\left(\sqrt{\hat{\rho}_K}\right), (y,z) \in \mathcal{G}_2\left(\sqrt{\hat{\rho}_K}\right)\right\}, \\
&\text{where } \; G_k^d(x,y)(t) \equiv \mu_k x(t) - \lambda_k m_k \Phi(x+y)(t).
\end{aligned} \tag{3.14}$$

For T_k, *if* $K > 1$, *then* $\mathcal{K}_{T_k} = \left\{G_k^d(y,z)/\mu_k : (y,z) \in \mathcal{G}_2\left(\sqrt{\hat{\rho}_K}\right)\right\}$, $k = 1, 2, \ldots, K$; *if* $K = 1$, *then* $\mathcal{K}_{T_1} = \left\{-\Psi(x) : x \in \mathcal{G}\left(c_1/\sqrt{\mu_1}\right)\right\}$.
(iii) *If* $\rho > 1$, *then the functional LIL in* (3.4) *holds with*

$$\mathcal{K}_k^* = \left\{\left(\sqrt{\hat{\rho}_K}x, \eta, q_k^* x, w_k^* x, t_k^* x, d_k^* x, \sqrt{\hat{\rho}_K}x\right) : x \in \mathcal{G}(1)\right\}, \tag{3.15}$$

where d_k, q_k, w_k *and* t_k *are defined in* (3.9)–(3.12).

We give the following remark to help readers to understand Theorem 3.2.

Remark 3.2 (Understanding the functional LIL limits in three cases) **(i) Workload** In the underloaded case, the stochastic variation of the workload is zero because the total workload is stochastically bounded. The functional LIL limits for queue length and workload are zero sets. In the critically loaded case, the stochastic variation of the total workload is the sum of those of all classes, in addition, we observe a functional LIL-version Little's Law for the queue length and workload, namely, $\mathcal{K}_{Q_k} = \mu_k \mathcal{K}_{W_k}$. The functional LIL limit of the departure process is given by a two-dimensional set, which is consistent with Iglehart (1971). Results become somewhat less intuitive in the overloaded case: First, the LIL of the total workload is not simply the aggregation of those of all classes, because stochastic fluctuations of workloads are not independent across classes. Moreover, the functional LIL-version Little's Law fails, unless all the squared coefficients of variation of service times are zeros. **(ii) Idle time** The variation of the idle time relates to busy times of all classes. For example, see the relation $\hat{\rho}_K = \sum_{k=1}^K \rho_k c_k^2/\mu_k$ given in (3.13) in the underloaded case (The relation is more complex in the critically loaded case; see (3.15)). In the overloaded case, the idle time is stochastically bounded for all servers and thus asymptotically negligible. This is in sharp contrast to the results of the busy times, where the stochastic variation of one class symmetrically affect all other classes. **(iii) Sojourn time** The stochastic variability of sojourn times is asymptotically independent with the class k ($1 \le k \le K$) and customer index i ($i = 1, 2, \ldots$), because the functional LIL limit of the sojourn times $\mathcal{K}_{\mathcal{T}_{k,i}}$ coincides with the functional LIL limit of the workload $\mathcal{K}_Z$ for all $i = 1, 2, \ldots$ and $1 \le k \le K$. This is so because customers are served under FCFS and the batch sizes become asymptotically negligible as the scale increases.

Next, we consider the $GI/GI/1$ queue, which is a special of $(GI^B/GI)^K/1$ with $K = 1$, $\xi_1(1) = 1$ and $c_1^2 = c_{a,1}^2 + c_{s,1}^2$. In the special case, we note that $\mathcal{T}_{1,1}(t) = Z(t) + v_1^*(1) = W(t) + v_1^*(1)$, and as a result, the corresponding limit sets satisfy $\mathcal{K}_Z = \mathcal{K}_{W_1} = \mathcal{K}_{\mathcal{T}_{1,1}}$ for $v_1^*(1) = o(\varphi(n))$ w.p.1 as $n \to \infty$. Hence, the functional LIL result in (3.4) simplifies to

$$\left(Z^n, Y^n, Q_1^n, T_1^n, D_1^n\right) \rightrightarrows \left(\mathcal{K}_Z, \mathcal{K}_Y, \mathcal{K}_{Q_1}, \mathcal{K}_{T_1}, \mathcal{K}_{D_1}\right) \equiv \mathcal{K}_-^*, \quad \text{w.p.1.} \tag{3.16}$$

The corollary below is proved by letting $m_k = 1$, $c_{b,k} = 0$, $K = 1$ in Theorem 3.2.

Corollary 3.1 (Functional LIL for the $GI/GI/1$ queue) *Suppose* (3.7) *holds,* $K = 1$ *and* $\xi_1(n) = 1$ *for all* $n \ge 1$. (i) *If* $\rho < 1$, *the functional LIL in* (3.16) *holds with*

$$\mathcal{K}_-^* = \left\{ \left(\eta, \frac{\lambda_1^{1/2} c_1}{\mu_1} x, \eta, \frac{\lambda_1^{1/2} c_1}{\mu_1} x, \lambda_1^{1/2} c_{1,a} x \right) : x \in \mathcal{G}(1) \right\}. \tag{3.17}$$

(ii) *If* $\rho = 1$, *then with* G_1^d *defined in* (3.14), *the functional LIL in* (3.16) *holds with*

$$\mathcal{K}_-^* = \left\{ \left(\Phi(x), \Psi(x), \mu_1 \Phi(x), -\Psi(x), G_1^d(y, z) \right) : x \in \mathcal{G}\left(\frac{c_1}{\sqrt{\mu_1}} \right), (y, z) \in \mathcal{G}_2\left(\frac{c_1}{\sqrt{\mu_1}} \right) \right\}. \tag{3.18}$$

(iii) *If* $\rho > 1$, *then the functional LIL in* (3.16) *holds with*

$$\mathcal{K}_-^* = \left\{ \left(\frac{\lambda^{1/2} c_1}{\mu_1} x, \eta, q_1^* x, \eta, d_1^* x \right) : x \in \mathcal{G}(1) \right\}, \tag{3.19}$$

where $q_1^* = \sqrt{\lambda_1 c_{a,1}^2 + \mu_1 c_{s,1}^2}$, $d_1^* = \sqrt{\mu_1 c_{s,1}^2}$.

In the overloaded case, it is evident from (3.19) that Little's law fails, that is, $\mu_1 \mathcal{K}_Z \ne \mathcal{K}_{Q_1}$; The asymptotical variation for departure does not depend on the arrival because the departure process is only determined by the service in the overloaded case. Next, we expect some sort of "seesaw" effect between the busy time T_1 and idle time I_1: $T_1(t)$ is negatively correlated with $I_1(t)$ at the same level (but different signs) of stochastic fluctuations (if $T_1(t)$ is larger than the mean value $\bar{T}_1(t)$ by t_0, caused by temporarily frequent arrivals or long service times, then $I_1(t)$ will be t_0 time less than $\bar{I}_1(t)$). This explains why $\mathcal{K}_{T_1} = \mathcal{K}_{I_1}$. Especially, in the overloaded case the server is almost busy all the time, the busy time deviates from its mean by a asymptotic finite value, which gives the functional LIL of the busy time is zero set, which also explains why $\mathcal{K}_{I_1} = \{\eta\}$.

We now give the LIL limits in (3.2) using input data $\mathcal{D}$ in (2.10).

Theorem 3.3 (LIL for the $(GI^B/GI)^K/1$ queue) *Suppose* (3.7) *holds.* (i) *If* $\rho < 1$, *then for* $k = 1, 2, \ldots, K$ *and* $i = 1, 2, \ldots$,

$$Z_{\sup}^* = Z_{\inf}^* = Q_{k,\sup}^* = Q_{k,\inf}^* = W_{k,\sup}^* = W_{k,\inf}^* = \mathcal{T}_{k,i,\sup}^* = \mathcal{T}_{k,i,\inf}^* = 0,$$
$$Y_{\sup}^* = -Y_{\inf}^* = \sqrt{\hat{\rho}_K}, \quad T_{k,\sup}^* = -T_{k,\inf}^* = \sqrt{\rho_k c_k^2 / \mu_k},$$

$$D^*_{k,\sup} = -D^*_{k,\inf} = m_k \lambda_k^{1/2} \sqrt{c^2_{a,k} + c^2_{b,k}}. \tag{3.20}$$

(ii) *If* $\rho = 1$*, then, for* $k = 1, 2, \ldots, K$ *and* $i = 1, 2, \ldots,$

$$\begin{aligned} Z^*_{\inf} = Y^*_{\inf} = \mathcal{T}^*_{k,i,\inf} = Q^*_{k,\inf} = W^*_{k,\inf} = 0, \\ \frac{\rho}{\lambda_k} Q^*_{k,\sup} = \frac{\rho}{\rho_k} W^*_{k,\sup} = Z^*_{\sup} = Y^*_{\sup} = \mathcal{T}^*_{k,i,\sup} = \sqrt{\hat{\rho}_K}. \end{aligned} \tag{3.21}$$

(iii) *If* $\rho > 1$*, then, for* $k = 1, 2, \ldots, K$ *and* $i = 1, 2, \ldots,$

$$\begin{aligned} &Z^*_{\sup} = -Z^*_{\inf} = \mathcal{T}^*_{k,i,\sup} = -\mathcal{T}^*_{k,i,\inf} = \sqrt{\hat{\rho}_K}, \quad Y^*_{\sup} = Y^*_{\inf} = 0, \quad D^*_{k,\sup} = -D^*_{k,\inf} = d^*_k, \\ &Q^*_{k,\sup} = -Q^*_{k,\inf} = q^*_k, \quad W^*_{k,\sup} = -W^*_{k,\inf} = w^*_k, \quad T^*_{k,\sup} = -T^*_{k,\inf} = t^*_k. \end{aligned} \tag{3.22}$$

Remark 3.3 (The Little's law) The superior and inferior limit version Little's law holds in the underloaded and critically loaded cases, that is, $Q^*_{k,\sup} = \mu_k W^*_{k,\sup}$ and $Q^*_{k,\inf} = \mu_k Z^*_{k,\inf}$, which means that the maximum and minimum asymptotical variations of Q_k and W_k keep the proportional relation. However, this version Little's law fails in the overloaded, because the workload process W_k keeps track of the total amount of unfinished service times of class k while the queue length process Q_k only counts the number of the unfinished class k customers. Although, as time goes to infinity, some class k customers will almost "never" be served in the overloaded case, their service variability will still make an impact to the workload: if the class k service times are highly variable, it does not affect its queue lengths because almost no one will enter service, but it will make the workload process highly variable because the customer's service time will be added to the workload process immediately upon its arrival. But the superior and inferior limit version Little's law holds in the critically loaded case if the coefficients of variation $c_{s,k} = 0$ by (3.10) and (3.11), which means that it is the variation of service times that make the LIL-version Little's law fail.

Corollary 3.2 (LIL for the $GI/GI/1$ queue) *Suppose* (3.7) *holds,* $K = 1$ *and* $\xi_1(n) = 1$ *for all* $n \geq 1$. (i) *If* $\rho < 1$*, then*

$$\begin{aligned} Z^*_{\sup} = Z^*_{\inf} = Q^*_{1,\sup} = Q^*_{1,\inf} = 0, \quad D^*_{1,\sup} = -D^*_{1,\inf} = \lambda_1^{1/2} c_{a,1}, \\ Y^*_{\sup} = -Y^*_{\inf} = T^*_{1,\sup} = -T^*_{1,\inf} = \frac{\lambda_1^{1/2} c_1}{\mu_1}. \end{aligned} \tag{3.23}$$

(ii) *If* $\rho = 1$*, then*

$$Z^*_{\sup} = Y^*_{\sup} = -T^*_{1,\inf} = \frac{\lambda_1^{1/2} c_1}{\mu_1}, \quad Q^*_{1,\sup} = \lambda_1^{1/2} c_1, \quad Z^*_{\inf} = Y^*_{\inf} = T^*_{1,\sup} = Q^*_{1,\inf} = 0. \tag{3.24}$$

(iii) *If* $\rho > 1$*, then*

$$\begin{aligned} Z^*_{\sup} = -Z^*_{\inf} = \frac{\lambda_1^{1/2} c_1}{\mu_1}, \quad Y^*_{\sup} = Y^*_{\inf} = T^*_{1,\sup} = -T^*_{1,\inf} = 0, \\ Q^*_{1,\sup} = -Q^*_{1,\inf} = \sqrt{\lambda_1 c^2_{a,1} + \mu_1 c^2_{s,1}}, \quad D^*_{1,\sup} = -D^*_{1,\inf} = \mu_1^{1/2} c_{s,1}. \end{aligned} \tag{3.25}$$

The LIL limits for queue, workload and departure qualify their asymptotic variation more distinct via number value not function set. In the underloaded case, the whole system is in light traffic, and all customers are quickly served upon arrival, which explains why the LIL limits for queue and workload are zero and for departure are independent of service. In the critically case, the LIL limits for queue and workload is influenced by both arrival and service, and satisfy the Little's law. In the overloaded case, the server is almost busy all the time, which explains why the departure LIL only depends on the service; the LIL limits of queue and workload capture the parameters of arrival and service, however, they deviate the Little's law, the reason refers to Remark 3.3.

The "seesaw" effect between the busy and idle times is presented by the LIL limits. In the underloaded case, the superior and inferior limits are presented as a pair of opposite numbers capturing the arrival and service parameters; In the critically case, since $T(t) \leq t$, we have $T^*_{1,\sup} < 0$ by the LIL scaling (3.1), however $T^*_{1,\sup} = 0$ explains that the server are almost busy in the critically heavy traffic; In the overloaded case, the server is busy all the time, which gives $T^*_{1,\sup} = T^*_{k,\inf} = Y^*_{\sup} = Y^*_{\inf} = 0$.

4 Proofs

In this section, we firstly prove Theorem 3.2 by Strassen's result (Strassen 1964) and the strong approximations (Chen and Shen 2000), then Theorem 3.3 with the help of the compact sets in Theorem 3.2.

4.1 Preliminaries

Strassen (1964) firstly developed the functional LIL for Brownian motion as follows.

Lemma 4.1 (Strassen's functional LIL) *Suppose that W_1, W_2 are two mutually independent one-dimensional standard Brownian motions, $\sigma_1 > 0$, $\sigma_2 > 0$ are two positive constants, we have, for any $t \in [0.1]$, w.p.1,*

$$\frac{\sigma_1 W_1(nt)}{\sqrt{2n \log \log n}} \rightrightarrows \mathcal{G}(\sigma_1), \quad \left(\frac{\sigma_1 W_1(nt)}{\sqrt{2n \log \log n}}, \frac{\sigma_2 W_2(nt)}{\sqrt{2n \log \log n}}\right) \rightrightarrows \mathcal{G}_2\left(\sqrt{\sigma_1^2 + \sigma_2^2}\right).$$

Our proof is mainly based on the strong approximation of all the performance measures. The strong approximations approximate a discrete process (i.e. Q) into a continuous process consisting of the deterministic fluid function (i.e. $\bar{Q}$) and standard Brownian motion. As a result, the original discrete system is approximated into a continuous Brownian system, whose mean values and fluctuation are embodied in the fluid limits and Brownian motion, respectively.

Lemma 4.2 presents the strong approximations for the queueing system, its proof is similar to Chen and Shen (2000) and is omitted.

Lemma 4.2 [Strong approximations (Chen and Shen 2000)] *If (3.7) holds, then, for $r > 2$, w.p.1,*

$$\begin{aligned} &||Z - \widetilde{Z}||_T = o(T^{1/r}), \quad ||Y - \widetilde{Y}||_T = o(T^{1/r}), \quad ||Q_k - \widetilde{Q}_k||_T = o(T^{1/r}), \\ &\qquad ||T_k - \widetilde{T}_k||_T = o(T^{1/r}) \\ &||W_k - \widetilde{W}_k||_T = o(T^{1/r}), \quad ||D_k - \widetilde{D}_k||_T = o(T^{1/r}), \quad ||\mathcal{T}_{k,i} - \widetilde{\mathcal{T}}_{k,i}||_T = o(T^{1/r}), \end{aligned} \tag{4.1}$$

for $k = 1, 2, \ldots, K$, $i = 1, 2, \ldots,$ where

$$\widetilde{Z} = \Phi(\widetilde{N}), \quad \widetilde{Y} = \Psi(\widetilde{N}), \quad \widetilde{T}_k(t) = \frac{1}{\mu_k}\left[\widetilde{D}_k(t) - \hat{S}_k(\bar{T}_k(t))\right],$$

$$
\begin{aligned}
\widetilde{N}(t) &= \bar{N}(t) + \sum_{k=1}^{K}\left[\frac{m_k}{\mu_k}\hat{A}_k(t) + \frac{1}{\mu_k}\hat{B}_k(\lambda_k t) + \hat{V}_k(m_k\lambda_k t)\right],\\
\widetilde{D}_k(t) &= \bar{D}_k(t) + m_k\hat{A}_k(\bar{\tau}(t)) + \hat{B}_k(\lambda_k\bar{\tau}(t)) - \frac{m_k\lambda_k}{\rho}[\widetilde{Y}(t) - \bar{Y}(t)]\\
&\quad -\frac{m_k\lambda_k}{\rho}\sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i(\bar{\tau}(t)) + \frac{1}{\mu_i}\hat{B}_i(\lambda_i\bar{\tau}(t)) + \hat{V}_i(\bar{D}_i(t))\right]\\
\widetilde{Q}_k(t) &= m_k\lambda_k t + m_k\hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \widetilde{D}_k(t), \quad \widetilde{T}_{k,i}(t) = \widetilde{Z}_k(t),\\
\widetilde{W}_k(t) &= \rho_k t + \frac{m_k}{\mu_k}\hat{A}_k(t) + \frac{1}{\mu_k}\hat{B}_k(\lambda_k t) + \hat{V}_k(m_k\lambda_k t) - \hat{V}_k(\bar{D}_k(t)) - \frac{1}{\mu_k}\widetilde{D}_k(t), \quad (4.2)
\end{aligned}
$$

$\hat{A}_k(t) = \lambda_k^{1/2}c_{a,k}W_{a,k}(t)$, $\hat{S}_k(t) = \mu_k^{1/2}c_{s,k}W_{s,k}(t)$, $\hat{B}_k(t) = m_k c_{b,k}W_{b,k}(t)$ and $\hat{V}_k(t) = -\hat{S}_k(t/\mu_k)/\mu_k$, and $W_{a,k}$, $W_{s,k}$ and $W_{b,k}$ are mutually independent Brownian motions associated with the arrival, service and batch processes for class k, respectively, and Φ and Ψ are defined in (2.8).

4.2 Proofs of Theorems 3.1, 3.2 and 3.3

Proof of Theorem 3.1 First of all, we note that $\mathsf{E}[B_k(A_k(t))] = m_k\lambda_k t$ for any $t \geq 0$. Next we first prove the functional LIL and then the LIL. By Lemma 2.3 (iv) in Chen and Shen (2000),

$$
\sup_{0\le t\le T}\left[B_k(A_k(t)) - m_k\lambda_k t - m_k\hat{A}_k(t) - \hat{B}_k(\lambda_k t)\right] = o(T^{1/r}), \quad \text{w.p.1},
$$

then for any $t \in [0, 1]$,

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{\left|B_k(A_k(nt)) - m_k\lambda_k nt - m_k\hat{A}_k(nt) - \hat{B}_k(\lambda_k nt)\right|}{\varphi(n)}\\
&\le \lim_{n\to\infty}\frac{\sup_{0\le t\le n}\left|B_k(A_k(t)) - m_k\lambda_k t - m_k\hat{A}_k(t) - \hat{B}_k(\lambda_k t)\right|}{\varphi(n)} = 0 \quad \text{w.p.1.} \qquad (4.3)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\frac{B_k(A_k(nt)) - m_k\lambda_k nt}{\varphi(n)}\\
&= \frac{B_k(A_k(nt)) - m_k\lambda_k nt - m_k\hat{A}_k(nt) - \hat{B}_k(\lambda_k nt)}{\varphi(n)} + \frac{m_k\hat{A}_k(nt) + \hat{B}_k(\lambda_k nt)}{\varphi(n)},
\end{aligned}
$$

and

$$
\frac{m_k\hat{A}_k(nt) + \hat{B}_k(\lambda_k nt)}{\varphi(n)} \rightrightarrows m_k\sqrt{\lambda_k(c_{a,k}^2 + c_{b,k}^2)}\mathcal{G}(1), \quad \text{w.p.1},
$$

then the functional LIL holds.

For the LIL, we note that $\sup_{x\in\mathcal{G}(1)} x(1) = 1$ and $\inf_{x\in\mathcal{G}(1)} x(1) = -1$, where the supremum and infimum are actually attained for the functions $x(t) = t$ and $x(t) = -t$ respectively. Hence, (3.8) holds. □

Proof of Theorem 3.2 For all $t \in [0, 1]$ and $n = 3, 4, \ldots,$ define

$$\widetilde{Z}^n(t) = \frac{\widetilde{Z}(nt) - \bar{Z}(nt)}{\varphi(n)}. \tag{4.4}$$

Similar we define $\widetilde{Y}^n(t)$, $\widetilde{N}^n(t)$, $\widetilde{Q}_k^n(t)$, $\widetilde{W}_k^n(t)$, $\widetilde{D}_k^n(t)$, $\widetilde{T}_k^n(t)$ and $\widetilde{\mathcal{T}}_{k,i}^n(t)$ in the same token of (4.4) for $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots$.

By Lemma 4.2, since $T^{1/r} = o(\varphi(T))$ for all $r > 2$, we have, for all $t \in [0, 1]$,

$$\lim_{n\to\infty} \frac{\left|Z(nt) - \widetilde{Z}(nt)\right|}{\varphi(n)} \le \limsup_{n\to\infty} \frac{\sup_{0\le t\le n}\left|Z(t) - \widetilde{Z}(t)\right|}{\varphi(n)} = 0, \quad \text{w.p.1.}$$

So, for all $t \in [0, 1]$,

$$\lim_{n\to\infty} \frac{Z(nt) - \widetilde{Z}(nt)}{\varphi(n)} = 0, \quad \text{w.p.1.} \tag{4.5}$$

Note that

$$Z^n(t) = \frac{Z(nt) - \widetilde{Z}(nt)}{\varphi(n)} + \widetilde{Z}^n(t),$$

This, and (4.5), implies that it suffices to prove $\widetilde{Z}^n \rightrightarrows \mathcal{K}_Z$ if one tries to prove $Z^n \rightrightarrows \mathcal{K}_Z$. Similar results hold for $Y^n(t)$, $Q_k^n(t)$, $W_k^n(t)$, $D_k^n(t)$, $T_k^n(t)$, $\mathcal{T}_{k,i}^n(t)$ for $k = 1, 2, \ldots, K$ and $i = 1, 2, \ldots$. In words, we transfer the original problem (3.4) into, w.p.1,

$$\left(\widetilde{Z}^n, \widetilde{Y}^n, \widetilde{Q}_k^n, \widetilde{W}_k^n, \widetilde{T}_k^n, \widetilde{D}_k^n, \widetilde{\mathcal{T}}_{k,i}^n\right) \rightrightarrows \mathcal{K}^*. \tag{4.6}$$

We firstly note that, for all $t \in [0, 1]$, w.p.1,

$$\widetilde{N}^n(t) = \frac{\sum_{i=1}^K \left[\frac{m_i}{\mu_i}\hat{A}_i(nt) + \frac{1}{\mu_i}\hat{B}_i(\lambda_i nt) + \hat{V}_i(m_i\lambda_i nt)\right]}{\varphi(n)} \rightrightarrows \sqrt{\hat{\rho}_K}\mathcal{G}(1), \tag{4.7}$$

because the variance

$$\operatorname{Var}\left(\sum_{i=1}^K \left[\frac{m_i}{\mu_i}\hat{A}_i(t) + \frac{1}{\mu_i}\hat{B}_i(\lambda_i t) + \hat{V}_i(m_i\lambda_i t)\right]\right) = \hat{\rho}_K t. \tag{4.8}$$

Next, we prove (4.6) in three cases: The underloaded, critically loaded and overloaded regimes.

The underloaded case We first note that if $\rho < 1$, then by (2.9), for any $t \ge 0$,

$$\bar{\mathbb{X}}_k(t) = (\eta, (1-\rho)t, (\rho-1)t, t, \eta, \eta, m_k\lambda_k t, \rho_k t, \eta). \tag{4.9}$$

For the functional LIL for Z, by (4.2) and (4.9), $\widetilde{Z}$ is reflected Brownian motion with negative drift $\rho - 1$, then it follows from Theorem 6.3 in Chen and Yao (2001) that $\sup_{0\le t\le T}|\widetilde{Z}(t)| = O(\log T)$ w.p.1. This implies that, for all $t \in [0, 1]$,

$$\lim_{n\to\infty} \widetilde{Z}^n(t) = \lim_{n\to\infty} \frac{\left|\widetilde{Z}(nt)\right|}{\varphi(n)} \le \lim_{n\to\infty} \frac{\sup_{0\le t\le n}\left|\widetilde{Z}(t)\right|}{\varphi(n)} = 0, \quad \text{w.p.1.}$$

So, $\widetilde{Z}^n(t) \rightrightarrows 0$ w.p.1. For the functional LIL of Y, by (4.2) and (4.7) we have

$$\widetilde{Y}^n(t) = \frac{\widetilde{Z}(nt) + \widetilde{N}(nt) - \bar{N}(nt)}{\varphi(n)} = \widetilde{Z}^n(t) + \widetilde{N}^n(t) \rightrightarrows \sqrt{\hat{\rho}_K}\mathcal{G}(1), \quad \text{w.p.1.}$$

For the functional LIL of D_k, by (4.2) and (4.9),

$$\widetilde{D}_k(t) - \bar{D}_k(t) = m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \frac{m_k \lambda_k}{\rho}[\widetilde{Y}(t) - \bar{Y}(t)] - \frac{m_k \lambda_k}{\rho}[\widetilde{N}(t) - \bar{N}(t)]$$
$$= m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \frac{m_k \lambda_k}{\rho}[\widetilde{Z}(t) - \bar{Z}(t)],$$

so, for all $t \in [0, 1]$, w.p.1.,

$$\widetilde{D}_k^n(t) = \frac{\widetilde{D}_k(nt) - \bar{D}_{nk}(t)}{\varphi(n)}$$
$$= \frac{m_k \hat{A}_k(nt) + \hat{B}_k(\lambda_k nt)}{\varphi(n)} - \frac{m_k \lambda_k}{\rho}\widetilde{Z}^n(t) \rightrightarrows m_k \lambda_k^{1/2} \sqrt{c_{a,k}^2 + c_{b,k}^2} \mathcal{G}(1).$$

For the functional LIL of Q_k, by (4.2) and (4.9),

$$\widetilde{Q}_k^n(t) = \frac{\widetilde{Q}_k(nt)}{\varphi(n)} = \frac{m_k \hat{A}_k(nt) + \hat{B}_k(\lambda_k nt)}{\varphi(n)} - \widetilde{D}_k^n(t) = \frac{m_k \lambda_k}{\rho}\widetilde{Z}^n(t) \rightrightarrows 0, \quad \text{w.p.1.}$$

For the functional LIL of W_k, by (4.2) and (4.9),

$$\widetilde{W}_k(t) = \rho_k t + \frac{m_k}{\mu_k}\hat{A}_k(t) + \frac{1}{\mu_k}\hat{B}_k(\lambda_k t) - \frac{1}{\mu_k}\widetilde{D}_k(t) = \frac{\rho_k}{\rho}\widetilde{Z}(t),$$

so, $\widetilde{W}_k^n(t) = \rho_k \widetilde{Z}^n(t)/\rho \rightrightarrows 0$ w.p.1. For the functional LIL of T_k, by (4.2) and (4.9),

$$\widetilde{T}_k(t) - \bar{T}_k(t) = \frac{1}{\mu_k}\left[\bar{D}_k(t) + m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \frac{m_k \lambda_k}{\rho}\widetilde{Z}(t) - \hat{S}_k(\bar{T}_k(t))\right] - \bar{T}_k(t)$$
$$= \frac{1}{\mu_k}\left[m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \frac{m_k \lambda_k}{\rho}\widetilde{Z}(t) - \hat{S}_k(\rho_k t)\right],$$

so, with (4.8),

$$\widetilde{T}_k^n(t) = \frac{1}{\mu_k}\frac{m_k \hat{A}_k(nt) + \hat{B}_k(\lambda_k nt) - \hat{S}_k(\rho_k nt)}{\varphi(n)} - \frac{\rho_k}{\rho}\widetilde{Z}^n(t) \rightrightarrows \sqrt{\frac{\rho_k c_k^2}{\mu_k}}\mathcal{G}(1), \quad \text{w.p.1,}$$

because $\widetilde{Z}^n(t) \to 0$ w.p.1. For the functional LIL of $\mathcal{T}_{k,i}$, by (4.2) and (4.9), $\widetilde{\mathcal{T}}_{k,i}^n(t) = \widetilde{Z}_k^n(t) \rightrightarrows 0, \quad$ w.p.1.

The critically loaded case If $\rho = 1$, then by Lemma 2.1,

$$\bar{\mathbb{X}}_k(t) = (\eta, \eta, \eta, t, \eta, \eta, m_k \lambda_k t, \rho_k t, \eta)\,. \tag{4.10}$$

For the functional LILs of Z and Y, by (4.2), (4.7) and (4.10), and the continuity of Φ and Ψ, we have, for all $t \in [0, 1]$, w.p.1, $\widetilde{Y}^n(t) = \Psi(\widetilde{N}^n)(t) \rightrightarrows \sqrt{\rho_K}\Psi(\mathcal{G}(1))$, $\widetilde{Z}^n(t) = \Phi(\widetilde{N}^n)(t) \rightrightarrows \sqrt{\rho_K}\Phi(\mathcal{G}(1))$. For the functional LIL of D_k, by (4.2) and (4.10),

$$\widetilde{D}_k(t) - \bar{D}_k(t)$$
$$= m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) - m_k \lambda_k \widetilde{Y}(t) - m_k \lambda_k \sum_{i=1}^{K}\left[\frac{m_i \hat{A}_i(t) + \hat{B}_i(\lambda_i t)}{\mu_i} + \hat{V}_i(m_i \lambda_i t)\right]$$
$$= m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) + m_k \lambda_k \inf_{0 \le s \le t}\left\{\sum_{i=1}^{K}\left[\frac{m_i \hat{A}_i(s) + \hat{B}_i(\lambda_i s)}{\mu_i} + \hat{V}_i(m_i \lambda_i s)\right]\right\}$$

$$
\begin{aligned}
&- m_k\lambda_k \sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i(t) + \frac{1}{\mu_i}\hat{B}_i(\lambda_i t) + \hat{V}_i(m_i\lambda_i t)\right] \\
&= \mu_k \frac{m_k\hat{A}_k(t) + \hat{B}_k(\lambda_k t)}{\mu_k} + m_k\lambda_k \inf_{0\le s\le t}\left\{\left[\frac{m_k\hat{A}_k(s) + \hat{B}_k(\lambda_k s)}{\mu_k}\right]\right. \\
&\left. + \sum_{i\neq k}\left[\frac{m_i\hat{A}_i(s) + \hat{B}_i(\lambda_i s)}{\mu_i} + \hat{V}_i(m_i\lambda_i s)\right] + \hat{V}_k(m_k\lambda_k s)\right\} \\
&- m_k\lambda_k\left\{\frac{m_k\hat{A}_k(t) + \hat{B}_k(\lambda_k t)}{\mu_k} + \sum_{i\neq k}\left[\frac{m_i\hat{A}_i(t) + \hat{B}_i(\lambda_i t)}{\mu_i} + \hat{V}_i(m_i\lambda_i t)\right] + \hat{V}_k(m_k\lambda_k t)\right\} \\
&= G_k^d\left(\frac{m_k\hat{A}_k(t) + \hat{B}_k(\lambda_k t)}{\mu_k}, \sum_{i\neq k}\left[\frac{m_i\hat{A}_i(t) + \hat{B}_i(\lambda_i t)}{\mu_i} + \hat{V}_i(m_i\lambda_i t)\right] + \hat{V}_k(m_k\lambda_k t)\right),
\end{aligned}
\tag{4.11}
$$

where G_k^d is defined in (3.14). By Lemma 4.2, for all $t \in [0, 1]$, w.p.1.,

$$
\left(\frac{\frac{m_k\hat{A}_k(nt)+\hat{B}_k(\lambda_k nt)}{\mu_k}}{\varphi(n)}, \frac{\sum_{i\neq k}\left[\frac{m_i\hat{A}_i(nt)+\hat{B}_i(\lambda_i nt)}{\mu_i} + \hat{V}_i(m_i\lambda_i nt)\right] + \hat{V}_k(m_k\lambda_k nt)}{\varphi(n)}\right) \rightrightarrows \mathcal{G}_2\left(\sqrt{\hat{\rho}_K}\right).
$$

Because G_k^d is a continuous mapping, we have, for all $t \in [0, 1]$,

$$
\begin{aligned}
\widetilde{D}_k^n(t) &= \frac{G_k^d\left(\frac{m_k\hat{A}_k(nt)+\hat{B}_k(\lambda_k nt)}{\mu_k}, \sum_{i\neq k}\left[\frac{m_i\hat{A}_i(nt)}{\mu_i} + \frac{\hat{B}_i(\lambda_i nt)}{\mu_i} + \hat{V}_i(m_i\lambda_i nt)\right] + \hat{V}_k(m_k\lambda_k nt)\right)}{\varphi(n)} \\
&\rightrightarrows G_k^d(\mathcal{G}_2(\sqrt{\hat{\rho}_K})), \quad \text{w.p.1.}
\end{aligned}
$$

For the functional LIL of Q_k, by (4.2) and (4.10),

$$
\begin{aligned}
\widetilde{Q}_k(t) &= m_k\lambda_k\widetilde{Y}(t) + m_k\lambda_k\sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i(t) + \frac{1}{\mu_i}\hat{B}_i(\lambda_i t) + \hat{V}_i(m_k\lambda_i t)\right] \\
&= m_k\lambda_k\left[\widetilde{Y}(t) + \widetilde{N}(t)\right] = m_k\lambda_k\widetilde{Z}(t).
\end{aligned}
$$

So, for all $t \in [0, 1]$, w.p.1., $\widetilde{Q}_k^n(t) = m_k\lambda_k\widetilde{Z}^n(t) \rightrightarrows m_k\lambda_k\Phi\left(\sqrt{\hat{\rho}_K}\mathcal{K}(1)\right)$. For the functional LIL of W_k, by (4.2) and (4.10),

$$
\begin{aligned}
\widetilde{W}_k(t) &= \rho_k t + \frac{m_k}{\mu_k}\hat{A}_k(t) + \frac{1}{\mu_k}\hat{B}_k(\lambda_k t) - \frac{1}{\mu_k}\widetilde{D}_k(t), \\
&= \frac{1}{\mu_k}\left[\bar{D}_k(t) + m_k\hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \widetilde{D}_k(t)\right] = \rho_k\left[\widetilde{Y}(t) + \widetilde{N}(t)\right] = \rho_k\widetilde{Z}(t),
\end{aligned}
$$

so, for all $t \in [0, 1]$, w.p.1., $\widetilde{W}_k^n(t) = \rho_k\widetilde{Z}^n(t) \rightrightarrows \rho_k\Phi\left(\sqrt{\hat{\rho}_K}\mathcal{G}(1)\right)$. For the functional LIL of T_k, by (4.2) and (4.10),

$$
\begin{aligned}
\widetilde{T}_k(t) - \bar{T}_k(t) &= \frac{1}{\mu_k}\left[\widetilde{D}_k(t) - \hat{S}_k(\bar{T}_k(t))\right] - \bar{T}_k(t) \\
&= -\rho_k\widetilde{Y}(t) - \rho_k\sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i(t) + \frac{1}{\mu_i}\hat{B}_i(\lambda_i t) + \hat{V}_i(m_i\lambda_i t)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{m_k\hat{A}_k(t)+\hat{B}_k(\lambda_k t)-\hat{S}_k(\rho_k t)}{\mu_k}\\
&=\rho_k \inf_{0\le s\le t}\left\{\sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i(s)+\frac{1}{\mu_i}\hat{B}_i(\lambda_i s)+\hat{V}_i(m_i\lambda_i s)\right]\right\}\\
&\quad-\rho_k\sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i(t)+\frac{1}{\mu_i}\hat{B}_i(\lambda_i t)+\hat{V}_i(m_i\lambda_i t)\right]\\
&\quad+\frac{1}{\mu_k}\left[m_k\hat{A}_k(t)+\hat{B}_k(\lambda_k t)-\hat{S}_k(\rho_k t)\right]\\
&=\rho_k \inf_{0\le s\le t}\left\{\sum_{i=1}^{K}\frac{1}{\mu_i}\left[m_i\hat{A}_i(s)+\hat{B}_i(\lambda_i s)-\hat{S}_i(\rho_i s)\right]\right\}\\
&\quad-\rho_k\sum_{i=1}^{K}\frac{1}{\mu_i}\left[m_i\hat{A}_i(t)+\hat{B}_i(\lambda_i t)-\hat{S}_i(\rho_i t)\right]+\frac{1}{\mu_k}\left[m_k\hat{A}_k(t)+\hat{B}_k(\lambda_k t)-\hat{S}_k(\rho_k t)\right]\\
&=\begin{cases}\frac{1}{\mu_k}G_k^d\left(\frac{m_k\hat{A}_k(t)+\hat{B}_k(\lambda_k t)-\hat{S}_k(\rho_k t)}{\mu_k},\sum_{i\ne k}\frac{m_i\hat{A}_i(t)+\hat{B}_i(\lambda_i t)-\hat{S}_i(\rho_i t)}{\mu_i}\right), & K>1,\\ \frac{1}{\mu_1}G_1^d\left(\frac{m_1\hat{A}_1(t)+\hat{B}_1(\lambda_1 t)-\hat{S}_1(\rho_1 t)}{\mu_1},0\right)=-\Psi\left(\frac{m_1\hat{A}_1(\cdot)+\hat{B}_1(\lambda_1\cdot)-\hat{S}_1(\cdot)}{\mu_1}\right)(t), & K=1.\end{cases}
\end{aligned}
\tag{4.12}
$$

where Φ and G_k^d is defined in (2.8) and (3.14) respectively. Notice that, for all $t\in[0,1]$,

$$
\frac{1}{\mu_1}\left[m_1\hat{A}_1(t)+\hat{B}_1(\lambda_1 t)-\hat{S}_1(t)\right]\rightrightarrows\mathcal{G}\left(\sqrt{c_1^2/\mu}\right),\quad \text{w.p.1},
$$

and for $K>1$, w.p.1.,

$$
\begin{aligned}
&\left(\frac{\frac{1}{\mu_k}\left[m_k\hat{A}_k(nt)+\hat{B}_k(\lambda_k nt)-\hat{S}_k(\rho_k nt)\right]}{\varphi(n)},\frac{\sum_{i\ne k}\frac{1}{\mu_i}\left[m_i\hat{A}_i(t)+\hat{B}_i(\lambda_i t)-\hat{S}_i(\rho_i t)\right]}{\varphi(n)}\right)\\
&\rightrightarrows\mathcal{G}_2\left(\sqrt{\hat{\rho}_K}\right),
\end{aligned}
$$

so, for all $t\in[0,1]$, if $K=1$, then

$$
\begin{aligned}
\widetilde{T}_1^n(t)&=-\Psi\left(\frac{1}{\mu_1}\left[m_1\hat{A}_1(\cdot)+\hat{B}_1(\lambda_1\cdot)-\hat{S}_1(\cdot)\right]\right)(t)\\
&\rightrightarrows-\Psi\left(\mathcal{G}\left(\sqrt{c_1^2/\mu}\right)\right),\quad\text{w.p.1},
\end{aligned}
$$

if $K>1$, then w.p.1., we have

$$
\begin{aligned}
\widetilde{T}_k^n(t)&=\frac{1}{\mu_k}\frac{G_k^d\left(\frac{m_k\hat{A}_k(nt)+\hat{B}_k(\lambda_k nt)-\hat{S}_k(\rho_k nt)}{\mu_k},\sum_{i\ne k}\frac{m_i\hat{A}_i(t)+\hat{B}_i(\lambda_i t)-\hat{S}_i(\rho_i t)}{\mu_i}\right)}{\varphi(n)}\\
&\rightrightarrows\frac{1}{\mu_k}G_k^d\left(\mathcal{G}_2\left(\sqrt{\hat{\rho}_K}\right)\right).
\end{aligned}
$$

For the functional LILs of $\mathcal{T}_{k,i}$, by (4.2) and (4.10), for all $t\in[0,1]$, w.p.1. we have $\widetilde{\mathcal{T}}_{k,i}^n(t)=\widetilde{Z}_k^n(t)\rightrightarrows\sqrt{\hat{\rho}_K}\,\Phi(\mathcal{G}(1))$.

The overloaded case If $\rho > 1$, then by (2.9), for any $t \geq 0$,

$$\bar{\mathbb{X}}_k(t)=\left((\rho-1)t,\, \eta,\, (\rho-1)t,\, \frac{t}{\rho},\, \left(1-\frac{1}{\rho}\right) m_k\lambda_k t,\, \left(1-\frac{1}{\rho}\right)\rho_k t,\, \frac{m_k\lambda_k t}{\rho},\, \frac{\rho_k t}{\rho},\, (\rho-1)t\right). \tag{4.13}$$

We first show the functional LIL of Y. By (4.2) and (4.13), since $\widetilde{N}$ is a Brownian motion with positive drift $(\rho-1)$. Then $\lim_{t\to\infty} \widetilde{N}(t)/t = +\infty$ w.p.1, and

$$\sup_{t\geq 0} \widetilde{Y}(t) = \sup_{t\geq 0} \Psi(\widetilde{N})(t) < +\infty \quad \text{and} \quad \widetilde{Y}^n(t) = \frac{\widetilde{Y}(nt)}{\varphi(n)} \leq \frac{\sup_{t\geq 0} \widetilde{Y}(t)}{\varphi(n)} \to 0, \quad \text{w.p.1}$$

as $n \to \infty$, for all $t \geq 0$. As a direct result, we have, for all $t \in [0, 1]$, $\widetilde{Y}^n(t) \rightrightarrows 0$. For the functional LIL of Z, by (4.2) and (4.13), $\widetilde{Z}(t) - \bar{Z}(t) = \widetilde{N}(t) - \bar{N}(t) + \widetilde{Y}(t)$, and with (4.7), for all $t \in [0, 1]$, w.p.1.

$$\widetilde{Z}^n(t) = \widetilde{N}^n(t) + \widetilde{Y}^n(t) \rightrightarrows \sqrt{\hat{\rho}_K}\, \mathcal{G}(1). \tag{4.14}$$

For the functional LIL of D_k, by (4.2) and (4.13),

$$\begin{aligned}
\widetilde{D}_k(t) - \bar{D}_k(t) &= m_k \hat{A}_k\left(\frac{t}{\rho}\right) + \hat{B}_k\left(\frac{\lambda_k t}{\rho}\right) - \frac{m_k\lambda_k}{\rho}\widetilde{Y}(t) \\
&\quad - \frac{m_k\lambda_k}{\rho}\sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i\left(\frac{t}{\rho}\right) + \frac{1}{\mu_i}\hat{B}_i\left(\frac{\lambda_i t}{\rho}\right) + \hat{V}_i\left(\frac{m_i\lambda_i t}{\rho}\right)\right] \\
&\equiv \widetilde{D}'_k(t) - \frac{m_k\lambda_k}{\rho}\widetilde{Y}(t),
\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}
\widetilde{D}'_k(t) &= \left(1-\frac{\rho_k}{\rho}\right)\left[m_k \hat{A}_k\left(\frac{t}{\rho}\right) + \hat{B}_k\left(\frac{\lambda_k t}{\rho}\right)\right] \\
&\quad - \frac{m_k\lambda_k}{\rho}\left\{\sum_{i\neq k}\frac{m_i\hat{A}_i\left(\frac{t}{\rho}\right) + \hat{B}_i\left(\frac{\lambda_i t}{\rho}\right)}{\mu_i} - \sum_{i=1}^{K}\frac{\hat{S}_i\left(\frac{\rho_i t}{\rho}\right)}{\mu_i}\right\}
\end{aligned}$$

is a driftless Brownian motion with variance parameter $(d_k^*)^2$ defined in (3.9). So, for all $t \in [0, 1]$, we have,

$$\widetilde{D}_k^n(t) = \frac{\widetilde{D}'_k(nt)}{\varphi(n)} - \frac{\lambda_k}{\rho}\widetilde{Y}^n(t) \rightrightarrows \mathcal{G}(d_k^*), \quad \text{w.p.1},$$

where the convergence holds because $\widetilde{Y}^n(t) \rightrightarrows 0$ w.p.1.

For the functional LIL of Q_k, by (4.2) and (4.13), $m_k\lambda_k t - \bar{Q}_k(t) = \bar{D}_k(t)$, and

$$\widetilde{Q}_k(t) - \bar{Q}_k(t) = m_k\hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \left[\widetilde{D}_k(t) - \bar{D}_k(t)\right] \equiv \widetilde{Q}'_k(t) + \frac{m_k\lambda_k}{\rho}\widetilde{Y}(t),$$

where

$$\begin{aligned}
\widetilde{Q}'_k(t) &= m_k\hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \Big\{m_k\hat{A}_k(\bar{\tau}(t)) + \hat{B}_k(\lambda_k\bar{\tau}(t)) \\
&\quad - \frac{m_k\lambda_k}{\rho}\sum_{i=1}^{K}\left[\frac{m_i}{\mu_i}\hat{A}_i(\bar{\tau}(t)) + \frac{1}{\mu_i}\hat{B}_i(\lambda_i\bar{\tau}(t)) + \hat{V}_i(\bar{D}_i(t))\right]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) - m_k \hat{A}_k\left(\frac{t}{\rho}\right) - \hat{B}_k\left(\frac{\lambda_k t}{\rho}\right) \\
&\quad + \frac{1}{\rho} m_k \lambda_k \sum_{i=1}^{K} \left[\frac{m_i}{\mu_i}\hat{A}_i\left(\frac{t}{\rho}\right) + \frac{1}{\mu_i}\hat{B}_i\left(\frac{\lambda_i t}{\rho}\right) + \hat{V}_i\left(\frac{m_i\lambda_i t}{\rho}\right)\right] \\
&= m_k \hat{A}_k(t) + \hat{B}_k(\lambda_k t) - \left\{-\frac{m_k\lambda_k}{\rho\mu_k}\left[m_k\hat{A}_k\left(\frac{t}{\rho}\right) + \hat{B}_k\left(\frac{\lambda_k t}{\rho}\right) - \hat{S}_k\left(\frac{\rho_k t}{\rho}\right)\right]\right. \\
&\quad + \left[m_k\hat{A}_k\left(\frac{t}{\rho}\right) + \hat{B}_k\left(\frac{\lambda_k t}{\rho}\right)\right] \\
&\quad \left. - \frac{m_k\lambda_k}{\rho}\sum_{i\neq k}\left[\frac{m_i}{\mu_i}\hat{A}_i\left(\frac{t}{\rho}\right) + \frac{1}{\mu_i}\hat{B}_i\left(\frac{\lambda_i t}{\rho}\right) - \frac{1}{\mu_i}\hat{S}_i\left(\frac{\rho_i t}{\rho}\right)\right]\right\} \\
&=_d \left[1 - \frac{1-\frac{\rho_k}{\rho}}{\sqrt{\rho}}\right]\left[m_k\hat{A}_k(t) + \sqrt{\lambda_k}\hat{B}_k(t)\right] - \frac{\rho_k\sqrt{\rho_k}}{\rho\sqrt{\rho}}\hat{S}_k(t) \\
&\quad + \frac{m_k\lambda_k}{\rho}\sum_{i\neq k}\left[\frac{m_i}{\mu_i}\hat{A}_i\left(\frac{t}{\rho}\right) + \frac{1}{\mu_i}\hat{B}_i\left(\frac{\lambda_i t}{\rho}\right) - \frac{1}{\mu_i}\hat{S}_i\left(\frac{\rho_i t}{\rho}\right)\right]
\end{aligned}
$$

is a driftless Brownian motion with variance parameter $(q_k^*)^2$ defined in (3.10). So, for all $t \in [0, 1]$,

$$
\widetilde{Q}_k^n(t) = \frac{\widetilde{Q}_k'(nt)}{\varphi(n)} + \frac{\lambda_k}{\rho}\widetilde{Y}^n(t) \rightrightarrows \mathcal{G}(q_k^*), \quad \text{w.p.1},
$$

where the convergence holds because $\widetilde{Y}^n(t) \rightrightarrows 0$ w.p.1.

For the functional LIL of W_k, by (4.2) and (4.13), $\rho_k t - \bar{W}_k(t) = \bar{D}_k(t)/\mu_k$, and

$$
\begin{aligned}
\widetilde{W}_k(t) - \bar{W}_k(t) &= \frac{m_k}{\mu_k}\hat{A}_k(t) + \frac{1}{\mu_k}\hat{B}_k(\lambda_k t) + \hat{V}_k(m_k\lambda_k t) \\
&\quad - \hat{V}_k\left(\frac{m_k\lambda_k t}{\rho}\right) - \frac{1}{\mu_k}\left[\widetilde{D}_k(t) - \bar{D}_k(t)\right] \\
&\equiv \widetilde{W}_k'(t) + \frac{\rho_k}{\rho}\widetilde{Y}(t),
\end{aligned}
$$

where as $\widetilde{Q}_k'(t)$,

$$
\begin{aligned}
\widetilde{W}_k'(t) &= \frac{m_k}{\mu_k}\hat{A}_k(t) + \frac{1}{\mu_k}\hat{B}_k(\lambda_k t) - \frac{1}{\mu_k}\hat{S}_k(\rho_k t) + \frac{1}{\mu_k}\hat{S}_k\left(\frac{\rho_k t}{\rho}\right) \\
&\quad - \frac{m_k}{\mu_k}\hat{A}_k\left(\frac{t}{\rho}\right) - \frac{1}{\mu_k}\hat{B}_k\left(\frac{\lambda_k t}{\rho}\right) \\
&\quad + \frac{\rho_k}{\rho}\left\{\frac{1}{\mu_k}\left[m_k\hat{A}_k\left(\frac{t}{\rho}\right) + \hat{B}_k\left(\frac{\lambda_k t}{\rho}\right) - \hat{S}_k\left(\frac{\rho_k t}{\rho}\right)\right]\right. \\
&\quad \left. + \sum_{i\neq k}\frac{1}{\mu_i}\left[m_i\hat{A}_i\left(\frac{t}{\rho}\right) + \hat{B}_i\left(\frac{\lambda_i t}{\rho}\right) - \hat{S}_i\left(\frac{\rho_i t}{\rho}\right)\right]\right\} \\
&= \frac{1}{\mu_k}\left[1 - \frac{1}{\sqrt{\rho}}\left(1 - \frac{\rho_k}{\rho}\right)\right]\left[m_k\hat{A}_k(t) + \sqrt{\lambda_k}\hat{B}_k(t) - \sqrt{\rho_k}\hat{S}_k(t)\right]
\end{aligned}
$$

$$+\frac{\rho_k}{\rho\sqrt{\rho}}\sum_{i\neq k}\frac{1}{\mu_i}\left[m_i\hat{A}_i(t)+\sqrt{\lambda_i}\hat{B}_i(t)-\sqrt{\rho_i}\hat{S}_i(t)\right],$$

is a driftless Brownian motion with variance parameter (w_k^*) defined in (3.11). So, for all $t\in[0,1]$, by $\widetilde{Y}^n(t)\rightrightarrows 0$ w.p.1, we have

$$\widetilde{W}_k^n(t)=\frac{\widetilde{W}'(nt)}{\varphi(n)}+\frac{\rho_k}{\rho}\widetilde{Y}^n(t)\rightrightarrows\mathcal{G}(w_k^*),\quad \text{w.p.1.}$$

For the functional LIL of T_k, by (4.2), (4.13) and (4.15), $\bar{T}_k(t)=\bar{D}_k(t)/\mu_k$, and

$$\widetilde{T}_k(t)-\bar{T}_k(t)=\frac{1}{\mu_k}\left[\widetilde{D}_k(t)-\bar{D}_k(t)\right]-\frac{1}{\mu_k}\hat{S}_k\left(\frac{\rho_k t}{\rho}\right)\equiv\widetilde{T}_k'(t)-\frac{\rho_k}{\rho}\widetilde{Y}(t),$$

where as $\widetilde{Q}_k'(t)$,

$$\begin{aligned}\widetilde{T}_k'(t)&=\frac{1}{\mu_k}\left[m_k\hat{A}_k(\bar{\tau}(t))+\hat{B}_k(\lambda_k\bar{\tau}(t))-\hat{S}_k\left(\frac{\rho_k t}{\rho}\right)\right]\\&\quad-\frac{m_k\lambda_k}{\mu_k\rho}\sum_{i=1}^{K}\left[\frac{m_i\hat{A}_i(\bar{\tau}(t))}{\mu_i}+\frac{\hat{B}_i(\lambda_i\bar{\tau}(t))}{\mu_i}+\hat{V}_i(\bar{D}_i(t))\right]\\&=\frac{1}{\mu_k}\left[m_k\hat{A}_k\left(\frac{t}{\rho}\right)+\hat{B}_k\left(\frac{\lambda_k t}{\rho}\right)-\hat{S}_k\left(\frac{\rho_k t}{\rho}\right)\right]\\&\quad-\frac{\rho_k}{\rho\mu_k}\left[m_k\hat{A}_k\left(\frac{t}{\rho}\right)+\hat{B}_k\left(\frac{\lambda_k t}{\rho}\right)-\hat{S}_k\left(\frac{\rho_k t}{\rho}\right)\right]\\&\quad-\frac{\rho_k}{\rho}\sum_{i\neq k}\frac{1}{\mu_i}\left[m_i\hat{A}_i\left(\frac{t}{\rho}\right)+\hat{B}_i\left(\frac{\lambda_i t}{\rho}\right)-\hat{S}_i\left(\frac{\rho_i t}{\rho}\right)\right]\\&=\frac{1}{\mu_k\sqrt{\rho}}\left(1-\frac{\rho_k}{\rho}\right)\left[m_k\hat{A}_k(t)+\sqrt{\lambda_k}\hat{B}_k(t)-\sqrt{\rho_k}\hat{S}_k(t)\right]\\&\quad-\frac{\rho_k}{\rho\sqrt{\rho}}\sum_{i\neq k}\frac{1}{\mu_i}\left[m_i\hat{A}_i(t)+\sqrt{\lambda_k}\hat{B}_i(t)-\sqrt{\rho_i}\hat{S}_i(t)\right]\end{aligned}$$

is a driftless Brownian motion with variance parameter $(t_k^*)^2$ defined in (3.12). So, for all $t\in[0,1]$, together with $\widetilde{Y}^n(t)\rightrightarrows 0$ w.p.1, we have $\widetilde{T}_k^n(t)=\frac{\widetilde{T}_k'(nt)}{\varphi(n)}-\frac{\rho_k}{\rho}\widetilde{Y}(nt)\rightrightarrows\mathcal{G}(t_k^*)$, w.p.1. For the functional LILs of $\mathcal{T}_{k,i}$, by (4.2), (4.13) and (4.14), $\widetilde{\mathcal{T}}_k^n(t)=\widetilde{Z}_k^n(t)\rightrightarrows\sqrt{\hat{\rho}_K}\,\mathcal{G}(1)$, w.p.1. □

Proof of Theorem 3.3 **Case** $\rho<1$. By (3.13), since $\mathcal{K}_Z=\mathcal{K}_{Q_k}=\mathcal{K}_{W_k}=\mathcal{K}_{\mathcal{T}_{k,i}}=0$, we have for $k=1,2,\ldots,K$ and $i=1,2,\ldots$, $Z_{\sup}^*=Z_{\inf}^*=Q_{k,\sup}^*=Q_{k,\inf}^*=W_{k,\sup}^*=W_{k,\inf}^*=\mathcal{T}_{k,i,\sup}^*=\mathcal{T}_{k,i,\inf}^*=0$. For Y, T_k and D_k, we firstly observe that $\sup_{x\in\mathcal{G}(\delta)}x(1)=\delta$ and $\inf_{x\in\mathcal{G}(\delta)}x(1)=-\delta$ for any $\delta>0$, where the supremum and infimum are actually attained for the functions $x(t)=\delta t$ and $x(t)=-\delta t$ respectively. This and (3.13) imply that (3.20) holds.

Case $\rho=1$. For some $\delta>0$, we have $\sup_{x\in\Phi(\mathcal{G}(\delta))}x(1)=\delta$ and $\inf_{x\in\Phi(\mathcal{G}(\delta))}x(1)=0$, where the supremum and the infimum are actually attained for the functions $x(t)=\delta t$ and $x(t)=0$ respectively. Taking $\delta=\sqrt{\hat{\rho}_K}$ yields $Z_{\sup}^*$ and $Z_{\inf}^*$ given in (3.21). As a direct result, we also get $Q_{k,\sup}^*$, $Q_{k,\inf}^*$, $W_{k,\sup}^*$, $W_{k,\inf}^*$ and $\mathcal{T}_{k,i,\sup}^*$, $\mathcal{T}_{k,i,\inf}^*$ given in (3.21).

For the LIL limits of Y, we first consider the set $\Psi(\mathcal{G}(\delta))$ for some $\delta > 0$. On the one hand, by (3.6) $|y(b)| \leq \delta\sqrt{b} \leq \delta$ for any $y \in \mathcal{G}(\delta)$ and $0 \leq b \leq 1$, then, $\sup_{x\in\Psi(\mathcal{G}(\delta))} x(1) = \sup_{y\in\mathcal{G}(\delta)} \sup_{0\leq s\leq 1}\{-y(s)\} = \delta$, where the supremum is attained for the function $y(s) = -s$. On the other hand, notice $y(0) = 0$ for any $y \in \mathcal{G}(\delta)$, then $\inf_{x\in\Psi(\mathcal{G}(\delta))} x(1) = \inf_{y\in\mathcal{G}(\delta)} \sup_{0\leq s\leq 1}\{-y(s)\} = 0$, where the infimum is attained for the function $y(s) = 0$. Taking $\delta = \sqrt{\hat{\rho}_K}$ yields $Y^*_{\sup}$ and $Y^*_{\inf}$ given in (3.21).

Case $\rho > 1$. The analysis for the case $\rho > 1$ is similar with the case $\rho < 1$. □

Proof of Corollary 3.2 We only show $T^*_{1,\sup}$ and $T^*_{1,\inf}$ in the critically loaded. As the proof of Theorem 3.3 $\sup_{x\in\Psi(\mathcal{G}(c_1/\sqrt{\mu_1}))} x(1) = c_1/\sqrt{\mu_1}$ and $\inf_{x\in\Psi(\mathcal{G}(c_1/\sqrt{\mu_1}))} x(1) = 0$, where the supremum and the infimum are actually attained for the functions $x(t) = c_1 t/\sqrt{\mu_1}$ and $x(t) = 0$ respectively. This, together with Theorem 3.2 (ii): $\mathcal{K}_{T_1} = \left\{-\Psi(x) : x \in \mathcal{G}\left(c_1/\sqrt{\mu_1}\right)\right\}$, implies that $T^*_{1,\sup} = 0$, $T^*_{1,\inf} = -\frac{\lambda_1^{1/2}}{\mu_1} c_1$. □

5 Numerical examples

We consider two numerical examples to obtain insights of the LIL limits in Theorem 3.3.

5.1 Sensitivity to the traffic intensity ρ

Example 5.1 (Discontinuity of the LIL limits in the traffic intensity) We consider the model $(GI^B/GI)^5/1/\text{FCFS}$ in Sect. 2, let the first-order parameters $\lambda_k = 0.2$ for $k = 2, 3, 4, 5$, $\mu_k = m_k = 1$ for $k = 1, 2, \ldots, 5$, and the second-order parameters $c_{a,k} = c_{b,k} = c_{s,k} = 1$ for all $k = 1, 2, \ldots, 5$, and increase λ_1 from zero. The idea is to increase the $\rho = \lambda_1 + 0.8$ from 0.8 so that we can walk through the underloaded, critically loaded and overloaded cases in Theorem 3.3. We plot the LIL limits in Fig. 1.

Example 5.1 show us that $\lambda_1 = 0.2$ (or $\rho = 1$) is a unique point of discontinuity of the LIL limits $Z^*_{\sup}$, $Q^*_{k,\sup}$, $W^*_{k,\sup}$, $Z^*_{\inf}$, $Q^*_{k,\inf}$, $W^*_{k,\inf}$ as functions of λ_1. More explicit numerical details of Example 5.1 is given to supplement Sect. 5.

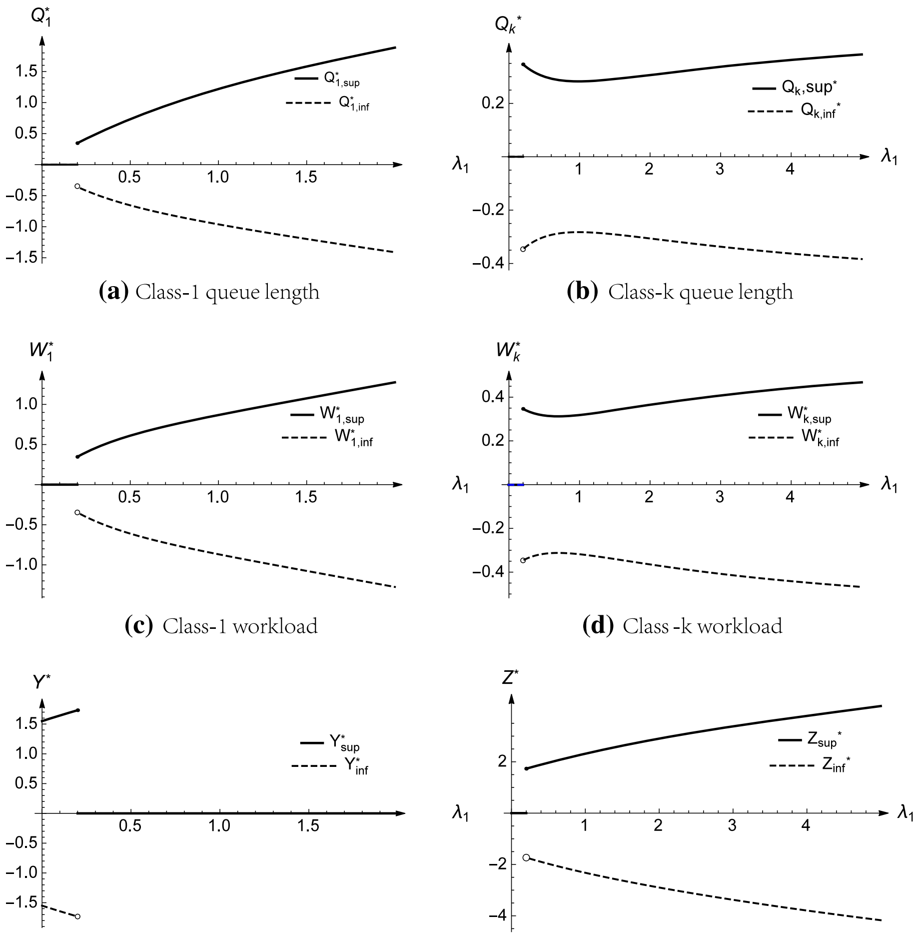
5.2 Impact from the second-order parameters

Example 5.2 (Impact of the LIL limits from the second-order parameters) We consider $(GI^B/GI)^5/1$ with the first-order parameters $m_k = \mu_k = 1$ and $c_{a,k} = c_a$, $c_{b,k} = c_b$, $c_{s,k} = c_s$ for all $k = 1, 2, \ldots, 5$. We try to see the dependence of the LIL limits in Theorem 3.3 on the second-order parameters c_a, c_b, c_s for $\lambda_1 = 0.1$ (underloaded), $\lambda_1 = 0.2$ (critically loaded) and $\lambda_1 = 0.4$ (overloaded) through varying one of c_a, c_b, c_s with others being fixed.

(i) The superior and inferior limits as functions of c_a are in Table 1.

$$q_1^* = \sqrt{\left[0.4\left(1 - \frac{2}{3\sqrt{1.2}}\right)^2 + \frac{2}{27}\right]} c_a,$$

$$q_k^* = \sqrt{\left[0.2\left(1 - \frac{5}{6\sqrt{1.2}}\right)^2 + \frac{1}{43.2}\right]} c_a, k \neq 1.$$



(a) Class-1 queue length **(b)** Class-k queue length

(c) Class-1 workload **(d)** Class -k workload

(e) Idel time **(f)** Total workload

Fig. 1 The LILs as functions of λ

(ii) The superior and inferior limits as functions of c_s are in Table 2.

$$w^*_{1,\mathrm{sup}} = \sqrt{\left(\frac{43}{27} - \frac{4}{3\sqrt{1.2}}\right) c_s}, \quad w^*_{k,\mathrm{sup}} = \sqrt{\left[\left(1 - \frac{5}{6\sqrt{1.2}}\right)^2 + \frac{5}{36}\right] c_s}, \, k \neq 1.$$

Remark 5.1 (For Example 5.2) (i) In Example 5.2, we do not show the impact from $c_{b,k}$ because it is similar with $c_{a,k}$. The corresponding numerical data are in Table 1 with c_a replaced by c_b. (ii) All the LIL limits are unary linear functions of second-order parameters c_a or c_s. (iii) The LIL-version Little's law, $Q^*_{k,\mathrm{sup}} = \mu_k W^*_{k,\mathrm{sup}}$ and $Q^*_{k,\mathrm{inf}} = \mu_k Z^*_{k,\mathrm{inf}}$ with $\mu_k = 1$, holds in the underloaded and critically loaded cases and fails in the overloaded case with $c_s \neq 0$, see Table 2. (iv) The LIL-version relation between workloads Z and W_k is clear in the underloaded and critically loaded cases by $Z^*_{\mathrm{sup}} = \sum_{k=1}^5 W^*_{k,\mathrm{sup}}$ and $Z^*_{\mathrm{inf}} = \sum_{k=1}^5 W^*_{k,\mathrm{inf}}$ and is unclear in the overloaded case. (v) The LIL-version relation between busy time T_k and idle time Y is clear in the underloaded case by $Y^{*2}_{\mathrm{sup}} = \sum_{k=1}^5 T^{*2}_{k,\mathrm{sup}}$ and $Y^{*2}_{\mathrm{inf}} = \sum_{k=1}^5 T^{*2}_{k,\mathrm{inf}}$

Table 1 The LIL limits as functions of c_a for $\lambda_1 = 0.1, 0.2, 0.4$ for Example 5.2

λ	$Z^*_{\sup}$	$Y^*_{\sup}$	$Q^*_{1,\sup}$	$Q^*_{k,\sup}$	$W^*_{1,\sup}$	$W^*_{k,\sup}$	$\mathcal{T}^*_{k,i,\sup}$	$T^*_{1,\sup}$	$T^*_{k,\sup}$	$D^*_{1,\sup}$	$D^*_{k,\sup}$
0.1	0	$\sqrt{0.9}c_a$	–	0	–	0	0	$\sqrt{0.1}c_a$	$\sqrt{0.2}c_a$	$\sqrt{0.1}c_a$	$\sqrt{0.2}c_a$
0.2	c_a	c_a	–	$0.2c_a$	–	$0.2c_a$	–	–	c_a	–	–
0.4	$\sqrt{1.2}c_a$	0	q^*_1	q^*_k	q^*_1	q^*_k	$\sqrt{1.2}c_a$	$\frac{2}{3}\sqrt{\frac{2}{3}}c_a$	$\frac{\sqrt{5}}{6}c_a$	$\frac{2}{3}\sqrt{\frac{2}{3}}c_a$	$\frac{\sqrt{5}}{6}c_a$
λ	$Z^*_{\inf}$	$Y^*_{\inf}$	$Q^*_{1,\inf}$	$Q^*_{k,\inf}$	$W^*_{1,\inf}$	$W^*_{k,\inf}$	$\mathcal{T}^*_{k,i,\inf}$	$T^*_{1,\inf}$	$T^*_{k,\inf}$	$D^*_{1,\inf}$	$D^*_{k,\inf}$
0.1	0	$-\sqrt{0.9}c_a$	–	0	–	0	0	$-\sqrt{0.1}c_a$	$-\sqrt{0.2}c_a$	$-\sqrt{0.1}c_a$	$-\sqrt{0.2}c_a$
0.2	0	0	–	0	–	0	–	–	0	–	–
0.4	$-\sqrt{1.2}c_a$	0	$-q^*_1$	$-q^*_k$	$-q^*_1$	$-q^*_k$	$-\sqrt{1.2}c_a$	$-\frac{2}{3}\sqrt{\frac{2}{3}}c_a$	$-\frac{\sqrt{5}}{6}c_a$	$-\frac{2}{3}\sqrt{\frac{2}{3}}c_a$	$-\frac{\sqrt{5}}{6}c_a$

Table 2 The LIL limits as functions of c_s for $\lambda_1 = 0.1, 0.2, 0.4$ for Example 5.2

λ	$Z^*_{\sup}$	$Y^*_{\sup}$	$Q^*_{1,\sup}$	$Q^*_{k,\sup}$	$W^*_{1,\sup}$	$W^*_{k,\sup}$	$\mathcal{T}^*_{k,i,\sup}$	$T^*_{1,\sup}$	$T^*_{k,\sup}$	$D^*_{1,\sup}$	$D^*_{k,\sup}$
0.1	0	$\sqrt{0.9}c_s$	–	0	–	0	0	$\sqrt{0.1}c_s$	$\sqrt{0.2}c_s$	–	0
0.2	c_s	c_s	–	$0.2c_s$	–	$0.2c_s$	–	–	c_s	–	–
0.4	$\sqrt{1.2}c_s$	0	$\frac{c_s}{\sqrt{3}}$	$\frac{\sqrt{13}c_s}{6\sqrt{6}}$	w^*_1	w^*_k	$\sqrt{1.2}c_s$	$\frac{\sqrt{2}}{3}c_s$	$\sqrt{\frac{5}{6}}c_s$	$\frac{1}{3}c_s$	$\frac{1}{2\sqrt{2}}c_s$
λ	$Z^*_{\inf}$	$Y^*_{\inf}$	$Q^*_{1,\inf}$	$Q^*_{k,\inf}$	$W^*_{1,\inf}$	$W^*_{k,\inf}$	$\mathcal{T}^*_{k,i,\inf}$	$T^*_{1,\inf}$	$T^*_{k,\inf}$	$D^*_{1,\inf}$	$D^*_{k,\inf}$
0.1	0	$-\sqrt{0.9}c_s$	–	0	–	0	0	$-\sqrt{0.1}c_s$	$-\sqrt{0.2}c_s$	–	0
0.2	0	0	–	0	–	0	–	–	0	–	–
0.4	$-\sqrt{1.2}c_s$	0	$-\frac{c_s}{\sqrt{3}}$	$-\frac{\sqrt{13}c_s}{6\sqrt{6}}$	$-w^*_1$	$-w^*_k$	$-\sqrt{1.2}c_s$	$-\frac{\sqrt{2}}{3}c_s$	$-\sqrt{\frac{5}{6}}c_s$	$-\frac{1}{3}c_s$	$-\frac{1}{2\sqrt{2}}c_s$

and is unclear in the overloaded case. (vi) The LIL-version relation between busy time T_k and departure D_k is proportional in the underloaded and the overloaded with $c_s = 0$, that is, $D^*_{k,\sup} = \mu_k T^*_{k,\sup}$ and $D^*_{k,\inf} = \mu_k T^*_{k,\inf}$ with $\mu_k = 1$ and is unclear in the overloaded case.

6 Conclusion

We have developed the functional LIL and its corresponding LIL in (1.1) for the $(GI^B/GI)^K/1$ queueing model for the total workload, idle time, queue length, workload, busy time, departure and sojourn time processes. Refining the functional strong law of large numbers and the corresponding limiting fluid functions which are often used to approximate the mean values, the functional LIL and its corresponding LIL provide estimates for the asymptotic rate of the increasing stochastic variability of these performance functions in two forms: the functional and numerical. We have identified these functional LIL and its corresponding LIL limits as explicit functional sets and functions of the first and second moments of the interarrival and service times of the primitive data. Our results, Theorems 3.2 and 3.3 , present all the functional LILs and its corresponding LIL limits covering all three regimes: the underloaded, the critically loaded, and the overloaded, categorized by the traffic intensity. For every case in the proof, we operate the strong approximation method in two steps: the first is for the functional LIL and the second is for the corresponding LIL by the obtained functional LIL sets.

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References

Borovkov, A. A. (1976). *Stochastic processes in queueing theory*. Berlin: Springer.

Borovkov, A. A. (1984). *Asymptotic methods in queueing theory*. New York: Wiley.

Caramellino, L. (1998). Strassens law of the iterated logarithm for diffusion processes for small time. *Stochastic Processes and Their Applications*, *74*(1), 1–19.

Chen, H., & Mandelbaum, A. (1994). Hierarchical modeling of stochastic network, part II: Strong approximations. In D. D. Yao (Ed.), *Stochastic modeling and analysis of manufacturing systems* (pp. 107–131). Berlin: Springer.

Chen, H., & Shen, X. (2000). Strong approximations for multiclass feedforward queueing networks. *Annals of Applied Probability*, *10*(3), 828–876.

Chen, H., & Yao, D. D. (2001). *Fundamentals of queueing networks*. New York: Springer.

Chen, H., & Zhang, H. Q. (1997). Stability of multiclass queueing networks under FIFO service discipline. *Mathematics of Operations Research*, *22*(3), 691–725.

Chen, H., & Zhang, H. Q. (2000). Diffusion approximations for some multiclass queueing networks with FIFO service disciplines. *Mathematics of Operations Research*, *25*(4), 679–707.

Clercq, S. D., Laevens, K., Steyaert, B., & Bruneel, H. (2013). A multi-class discrete-time queueing system under the FCFS service discipline. *Annals of Operations Research*, *202*(1), 59–73.

Csörgő, M., Deheuvels, P., & Horváth, L. (1987). An approximation of stopped sums with applications in queueing theory. *Advances in Applied Probability*, *19*(3), 674–690.

Csörgő, M., & Horváth, L. (1993). *Weighted approximations in probability and statistics*. New York: Wiley.

Csörgő, M., & Révész, P. (1981). *Strong approximations in probability and statistics*. New York: Academic Press.

Dai, J. G. (1995). On the positive Harris recurrence for multiclass queueing networks: A unified approach via fluid limit models. *Annals of Applied Probability*, *5*(1), 49–77.

Ethier, S. N., & Kurtz, T. G. (1986). *Markov processes: Characterization and convergence*. New York: Wiley.

Falin, G. (1994). The $M^K/G/\infty$ batch arrival queue with heterogeneous dependent demands. *Journal of Applied Probability*, *31*, 841–846.

Glynn, P. W., & Whitt, W. (1986). A central-limit-theorem version of $L = \lambda W$. *Queueing Systems*, *1*(2), 191–215.

Glynn, P. W., & Whitt, W. (1987). Sufficient conditions for functional limit theorem versions of $L = \lambda W$. *Queueing Systems*, *1*(3), 279–287.

Glynn, P. W., & Whitt, W. (1988). An LIL version of $L = \lambda W$. *Mathematics of Operations Research*, *13*(4), 693–710.

Glynn, P. W., & Whitt, W. (1991a). A new view of the heavy-traffic limit for infinite-server queues. *Advances in Applied Probability*, *23*(1), 188–209.

Glynn, P. W., & Whitt, W. (1991b). Departures from many queues in series. *Annals of Applied Probability*, *1*(4), 546–572.

Grigelionis, B., & Mikulevičius, R. (1987). Functional limit theorems for queueing systems in heavy traffic I, II. *Lietuvos Matematikos Rinkinys*, *27*(660–673), 441–454.

Guo, Y., & Liu, Y. (2015). A law of iterated logarithm for multiclass queues with preemptive priority service discipline. *Queueing System*, *79*(3), 251–291.

Guo, Y., Liu, Y., & Pei, R. (2018). Functional law of iterated logarithm for multi-server queues with batch arrivals and customer feedback. *Annals of Operations Research*, *264*, 157–191.

Harrison, J. M. (1985). *Brownian motion and stochastic flow system*. New York: Wiley.

Iglehart, D. L. (1971). Multiple channel queues in heavy traffic: IV. Law of the iterated logarithm. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, *17*, 168–180.

Iglehart, D. L., & Whitt, W. (1970a). Multiple channel queues in heavy traffic I. *Advances in Applied Probability*, *2*(1), 150–177.

Iglehart, D. L., & Whitt, W. (1970b). Multiple channel queues in heavy traffic II: Sequences, networks, and batches. *Advances in Applied Probability*, *2*(2), 355–369.

Horváth, L. (1984a). Strong approximation of renewal processes. *Stochastic Processes and Their Applications*, *18*(1), 127–138.

Horváth, L. (1984b). Strong approximation of extended renewal processes. *The Annals of Probability*, *12*(4), 1149–1166.

Horváth, L. (1992). Strong approximations of open queueing networks. *Mathematics of Operations Research*, *17*(2), 487–508.

Karpelevich, F. I., & Kreinin, A. Y. (1981). Two-phase queuing system $GI/G/1 \to G'/1/\infty$ under heavy traffic conditions. *Theory of Probability and Its Applications*, *26*(2), 293–313.

Kingman, J. F. C. (1962). On queues in heavy traffic. *Journal of Royal Statistical Society Series B*, *24*, 383–392.

Kulkarni, V. G., & Glazebrook, K. D. (2002). Output analysis of a single-buffer multiclass queue: FCFS service. *Journal of Applied Probability*, *39*(2), 341–358.

Lee, H. S., & Srinivasan, M. M. (1989). Control policies for $M^X/G/1$ queueing system. *Management Science*, *35*(6), 708–721.

Lévy, P. (1937). *Théorie de l'addition des variables aléatories*. Paris: Gauthier-Villars.

Lévy, P. (1948). *Procesus stochastique et mouvement Brownien*. Paris: Gauthier-Villars.

Liu, L., & Templeton, J. G. C. (1993). Autocorrelations in infinite server batch arrival queues. *Queueing Systems*, *14*, 313–337.

Mandelbaum, A., & Massey, W. A. (1995). Strong approximations for time-dependent queues. *Mathematics of Operations Research*, *20*(1), 33–64.

Mandelbaum, A., Massey, W. A., & Reiman, M. (1998). Strong approximations for Markovian service networks. *Queueing Systems*, *30*, 149–201.

Minkevičius, S. (2014). On the law of the iterated logarithm in multiserver open queueing networks. *Stochastics*, *86*(1), 46–59.

Minkevičius, S., & Steišūnas, S. (2003). A law of the iterated logarithm for global values of waiting time in multiphase queues. *Statistics and Probability Letters*, *61*(4), 359–371.

Pang, G. D., Talreja, R., & Whitt, W. (2007). Martingale proofs of many-server heavy-traffic limits for Markovian queues. *Probability Surveys*, *4*, 193–267.

Pang, G. D., & Whitt, W. (2012). Infinite-server queues with batch arrivals and dependent service times. *Probability in the Engineering and Information Sciences*, *26*, 197–220.

Prohorov, Y. V. (1963). Transition phenomena in queueing theory. *Lithuanian Mathematical Journal*, *3*(1), 199–206.

Sakalauskas, L. L., & Minkevičius, S. (2000). On the law of the iterated logarithm in open queueing networks. *European Journal of Operational Research*, *120*(3), 632–640.

Strassen, V. (1964). An invariance principle for the law of the iterated logarithm. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, *3*(3), 211–226.

Tsai, T. H. (2000). Empirical law of the iterated logarithm for Markov chains with a countable state space. *Stochastic Processes and Their Applications*, *89*(2), 175–191.

Van Ommeren, J. C. W. (1990). Simple approximations for the batch-arrival $M^X/G/1$ queue. *Operations Research*, *38*(4), 678–685.

Whitt, W. (1983a). The queueing network analyzer. *Bell System Technical Journal*, *62*(9), 2779–2815.

Whitt, W. (1983b). Comparing batch delays and customer delays. *Bell System Technical Journal*, *62*(7), 2001–2009.

Whitt, W., & Talreja, R. (2008). Fluid models for overloaded multiclass many-server queueing systems with FCFS routing. *Management Science*, *54*(8), 1513–1527.

Zhang, H. Q. (1997). Strong approximations of irreducible closed queueing networks. *Advances in Applied Probability*, *29*(2), 498–522.

Zhang, H. Q., & Hsu, G. X. (1992). Strong approximations for priority queues: Head-of-the-line-first discipline. *Queueing Systems*, *10*(3), 213–234.

Zhang, H. Q., Hsu, G. X., & Wang, R. X. (1990). Strong approximations for multiple channels in heavy traffic. *Journal of Applied Probability*, *27*(3), 658–670.