Appendix

EC.1. Proof of Lemma 3

We prove the second inequality as an illustration. Because there is only a finite number of subsets of $\mathcal{I}$, it is enough to prove that

$$\sum_{i\in\mathcal{I}_*(t)} \nu_i^* \bar{L}_i'(t) \geq \epsilon$$

for some ϵ which depends on $\mathcal{I}_*(t)$.

Denote by $\mathcal{J}_*(t)$ the subset of $\mathcal{J}$ such that $\mathcal{C}(j)\cap\mathcal{I}_*(t)\neq\emptyset$ for $j\in\mathcal{J}_*(t)$. Then $i\in\mathcal{I}_*(t)$ and $j\in\mathcal{S}(i)$ implies $i\in\mathcal{I}_*(t)$ and $j\in\mathcal{J}_*(t)$. Also, $i\in\mathcal{I}_*(t)$ and $j\in\mathcal{J}_*(t)\backslash\mathcal{S}(i)$ implies $\mu_{ij}=0$. As a result, we can write

$$\sum_{i\in\mathcal{I}_*(t)} \nu^* \bar{L}_i'(t) = \sum_{i\in\mathcal{I}_*(t)} \nu_i^*\lambda_i^e - \sum_{i\in\mathcal{I}_*(t)} \nu_i^* \sum_{j\in\mathcal{S}(i)} \mu_{ij}\bar{T}_{ij}'(t) = \sum_{i\in\mathcal{I}_*(t)} \nu_i^*\lambda_i^e - \sum_{i\in\mathcal{I}_*(t)} \nu_i^* \sum_{j\in\mathcal{J}_*(t)} \mu_{ij}\bar{T}_{ij}'(t).$$

From (9) and (17),

$$\begin{aligned}
\sum_{i\in\mathcal{I}_*(t)} \nu_i^* \sum_{j\in\mathcal{J}_*(t)} \mu_{ij}\bar{T}_{ij}'(t) &= \sum_{j\in\mathcal{J}_*(t)} \sum_{i\in\mathcal{I}_*(t)} \nu_i^*\mu_{ij}\bar{T}_{ij}'(t)\\
&= \sum_{j\in\mathcal{J}_*(t)} z_j^* \sum_{i\in\mathcal{I}_*(t)} \bar{T}_{ij}'(t)\\
&< \sum_{j\in\mathcal{J}_*(t)} z_j^* \sum_{i\in\mathcal{I}_*(t)} x_{ij}^*\\
&= \sum_{j\in\mathcal{J}_*(t)} \sum_{i\in\mathcal{I}_*(t)} \nu_i^*\mu_{ij}x_{ij}^*\\
&\leq \sum_{i\in\mathcal{I}_*(t)} \sum_{j\in\mathcal{S}(i)} \nu_i^*\mu_{ij}x_{ij}^*\\
&= \sum_{i\in\mathcal{I}_*(t)} \nu_i^* \sum_{j\in\mathcal{S}(i)} \mu_{ij}x_{ij}^* = \sum_{i\in\mathcal{I}_*(t)} \nu_i^*\lambda_i^e.
\end{aligned}$$

This strict inequality is because, for $j\in\mathcal{J}_*(t)$, if $\mathcal{C}(j)\cap(\mathcal{I}_*(t))^c=\emptyset$, then $\sum_{i\in\mathcal{I}_*(t)}\bar{T}_{ij}'(t)\leq 1=\sum_{i\in\mathcal{I}_*(t)} x_{ij}^*$; and there exists at least one j, such that $\mathcal{C}(j)\cap(\mathcal{I}_*(t))^c\neq\emptyset$; for such j, from (36), $\sum_{i\in\mathcal{I}_*(t)}\bar{T}_{ij}'(t)=0<\sum_{i\in\mathcal{I}_*(t)} x_{ij}^*$. □

EC.2. Proof of Proposition 1

We first prove (a). For ease of reference, we list the four steps for proving (a) again:

Step 1: There exists a finite $t_1\geq t_0$, such that for all $t\geq t_1$, $\mathcal{I}^*(t)\neq\{2\}$;

Step 2: For $t\geq t_1$, such that $\bar{Q}(t)$ is not a fixed point, $\sum_{i=1}^{3}{}_*\bar{Q}_i'(t)\geq\epsilon_1$ for some $\epsilon_1>0$;

Step 3: For all $t\geq t_0$, $\bar{W}(t)=\bar{W}(t_0)$;

Step 4: Assume that $\bar{Q}(t_2)$ is a fixed point, then for all $t\geq t_2$, $\bar{Q}(t)=\bar{Q}(t_2)=q^*(\bar{W}(t_0))$.

Proof of Step 1: If $\mathcal{I}^*(t_0) \neq \{2\}$, let $t_1 = t_0$; otherwise, note that if $\mathcal{I}^*(t) = \{2\}$, then from (36) and (35), $\bar{T}'_{11}(t) = \bar{T}'_{32}(t) = 0$ and $\bar{T}'_{21}(t) = \bar{T}'_{22}(t) = 1$. As a result,

$$\bar{Q}'_1(t) \geq 0 \quad \text{and} \quad \bar{Q}'_3(t) \geq 0,$$

and

$$\begin{aligned}\bar{Q}'_2(t) &= \lambda_2 - (1 - P_{22})(\mu_{21} + \mu_{22}) \\ &< \lambda_2 + P_{12}\mu_{11}x^*_{11} + P_{32}\mu_{32}x^*_{32} - (1 - P_{22})(\mu_{21}x^*_{21} + \mu_{22}x^*_{22}) \\ &= 0,\end{aligned}$$

thus, starting from t_0, $\bar{Q}_2$ decreases while $\bar{Q}_1$ and $\bar{Q}_3$ are nondecreasing. As a result, in a finite time (which can be bounded), say at time t_1, $\mathcal{I}^*(t_1) \neq \{2\}$.

Next assume there is a $t \geq t_1$, such that $\mathcal{I}^*(t) = \{2\}$. Denote by $\epsilon := \frac{C'_2(\bar{Q}_2(t))}{y^*_2} - \max_{i=1,3} \frac{C'_i(\bar{Q}_i(t))}{y^*_i} > 0$, then due to the continuity, and from the fact that $\frac{C'_2(\bar{Q}_2(t_1))}{y^*_2} \leq \max_{i=1,3} \frac{C'_i(\bar{Q}_i(t_1))}{y^*_i}$, there exists a time $s_0 \in [t_1, t]$ such that $\frac{C'_2(\bar{Q}_2(s_0))}{y^*_2} - \max_{i=1,3} \frac{C'_i(\bar{Q}_i(s_0))}{y^*_i} = \frac{\epsilon}{2}$ and for all $s \in [s_0, t]$, $\frac{C'_2(\bar{Q}_2(s))}{y^*_2} - \max_{i=1,3} \frac{C'_i(\bar{Q}_i(s))}{y^*_i} \geq \frac{\epsilon}{2}$. However, similar to the argument above, $\frac{C'_2(\bar{Q}_2(s))}{y^*_2}$ decreases on $[s_0, t]$ and $\max_{i=1,3} \frac{C'_i(\bar{Q}_i(s))}{y^*_i}$ does not decrease. As a result, one cannot get $\frac{C'_2(\bar{Q}_2(t))}{y^*_2} - \max_{i=1,3} \frac{C'_i(\bar{Q}_i(t))}{y^*_i} = \epsilon$. Hence, we must have $\mathcal{I}^*(t) \neq \{2\}$ for all $t \geq t_1$.

Proof of Step 2: First note that $\bar{Q}_i(t) \geq {}_*\bar{Q}_i(t)$ and $\bar{Q}_i(t) \leq {}^*\bar{Q}_i(t)$. If $i \in \mathcal{I}_*(t)$, then $\bar{Q}_i(t) = {}_*\bar{Q}_i(t)$, hence $\bar{Q}'_i(t) = {}_*\bar{Q}'_i(t)$. Similarly, if $i \in \mathcal{I}^*(t)$, then $\bar{Q}'_i(t) = {}^*\bar{Q}'_i(t)$. Finally note that all ${}_*\bar{Q}'_i$ have the same sign, and all ${}^*\bar{Q}'_i$ have the same sign.

Because $\bar{Q}(t)$ is not a fixed point and $I^*(t) \neq \{2\}$, we know that $\mathcal{I}_*(t)$ can be one of these: $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$ and $\{2,3\}$. We discuss these sets case-by-case.

1. $\mathcal{I}_*(t) = \{2\}$: from (35) and (36), we have $\bar{T}'_{11}(t) = 1$, $\bar{T}'_{32}(t) = 1$, $\bar{T}'_{21}(t) = 0$ and $\bar{T}'_{22}(t) = 0$, hence

$$\begin{aligned}{}_*\bar{Q}'_2(t) =& \bar{Q}'_2(t) = \lambda_2 + P_{12}\mu_{11} + P_{32}\mu_{32} \\ &> \lambda_2 + P_{12}\mu_{11}x^*_{11} + P_{32}\mu_{32}x^*_{32} - (1 - P_{22})(\mu_{21}x^*_{21} + \mu_{32}x^*_{32}) = 0.\end{aligned}$$

Because all ${}_*\bar{Q}'_i(t)$ have the same sign, then

$$\sum_{i=1}^{3} {}_*\bar{Q}'_i(t) \geq {}_*\bar{Q}'_2(t) > 0.$$

2. $\mathcal{I}_*(t) = \{1\}$ or $\mathcal{I}_*(t) = \{3\}$. We use $\mathcal{I}_*(t) = \{1\}$ to illustrate. There are two subcases:
   (a) $\mathcal{I}^*(t) = \{3\}$:

from (35) and (36), we have $\bar{T}'_{21}(t) = 1$, $\bar{T}'_{32}(t) = 1$, $\bar{T}'_{11}(t) = 0$ and $\bar{T}'_{22}(t) = 0$, hence

$${}_*\bar{Q}'_1(t) = \bar{Q}'_1(t) = \lambda_1 + P_{31}\mu_{32} + P_{21}\mu_{21} > 0,$$

the last strict inequality is because at least one of these three terms is positive. Then $\sum_{i=1}^{3} {}_*\bar{Q}'_i(t) \geq {}_*\bar{Q}'_1(t) > 0$.

(b) $\mathcal{I}^*(t)=\{2,3\}$:

First note that from (36), $\bar{Q}_1'(t)\geq 0$. Also note that $\bar{Q}_2'(t)$ and $\bar{Q}_3'(t)$ have the same sign because $\bar{Q}_2'(t)={}^*\bar{Q}_2'(t)$ and $\bar{Q}_3'(t)={}^*\bar{Q}_3'(t)$, and ${}^*\bar{Q}_2'(t)$ has the same sign as ${}^*\bar{Q}_3'(t)$. From the first inequality in Lemma 3 ($\sum_{i\in\mathcal{I}^*(t)}\nu_i^*\bar{L}_i'(t)\leq -\epsilon_1$), we conclude that both $\bar{Q}_2'(t)$ and $\bar{Q}_3'(t)$ are negative. From the second inequality in Lemma 3 ($\sum_{i\in\mathcal{I}_*(t)}\nu_i^*\bar{L}_i'(t)\geq \epsilon_1$), then $\bar{Q}_1'(t)$ is lower bounded by a strictly positive constant. Then $\sum_{i=1}^3 {}_*\bar{Q}_i'(t)\geq {}_*\bar{Q}_1'(t)=\bar{Q}_1'(t)>0$.

3. $\mathcal{I}_*(t)=\{1,2\}$ or $\mathcal{I}_*(t)=\{2,3\}$. We use $\mathcal{I}_*(t)=\{1,2\}$ to illustrate.

From (35) and (36), $\bar{T}_{32}'(t)=1$, $\bar{T}_{22}'(t)=0$ and $\bar{T}_{11}'(t)+\bar{T}_{21}'(t)=1$. For $\bar{Q}_1$ and $\bar{Q}_2$:

$$\begin{aligned}\bar{Q}_1'(t)=&\lambda_1+P_{31}\mu_{32}+P_{21}\mu_{21}\bar{T}_{21}'(t)-(1-P_{11})\mu_{11}\bar{T}_{11}'(t),\\ \bar{Q}_2'(t)=&\lambda_2+P_{32}\mu_{32}+P_{12}\mu_{11}\bar{T}_{11}'(t)-(1-P_{22})\mu_{21}\bar{T}_{21}'(t).\end{aligned}$$

From (35), $\bar{T}_{11}'(t)+\bar{T}_{21}'(t)=1$. Then (using (9))

$$(\nu_1^*,\nu_2^*)\cdot\begin{pmatrix}1-P_{11} & -P_{21}\\ -P_{12} & 1-P_{22}\end{pmatrix}^{-1}\begin{pmatrix}\bar{Q}_1'(t)\\ \bar{Q}_2'(t)\end{pmatrix}=(\nu_1^*,\nu_2^*)\cdot\begin{pmatrix}1-P_{11} & -P_{21}\\ -P_{12} & 1-P_{22}\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1+P_{31}\mu_{32}\\ \lambda_2+P_{32}\mu_{32}\end{pmatrix}-z_1^*.$$

Note that

$$\begin{aligned}&\lambda_1+P_{21}(\mu_{21}x_{21}^*+\mu_{22}x_{22}^*)+P_{31}\mu_{32}x_{32}^*-(1-P_{11})\mu_{11}x_{11}^*=0,\\ &\lambda_2+P_{12}\mu_{11}x_{11}^*+P_{32}\mu_{32}x_{32}^*-(1-P_{22})(\mu_{21}x_{21}^*+\mu_{22}x_{22}^*)=0.\end{aligned}$$

Thus

$$\begin{aligned}&\lambda_1+P_{31}\mu_{32}+P_{21}\mu_{21}x_{21}^*-(1-P_{11})\mu_{11}x_{11}^*=P_{31}\mu_{32}(1-x_{32}^*)-P_{21}\mu_{22}x_{22}^*,\\ &\lambda_2+P_{32}\mu_{32}+P_{12}\mu_{11}x_{11}^*-(1-P_{22})\mu_{21}x_{21}^*=P_{32}\mu_{32}(1-x_{32}^*)+(1-P_{22})\mu_{22}x_{22}^*.\end{aligned}$$

Thus, using (9) and (16) ($x_{11}^*+x_{21}^*=1$),

$$\begin{aligned}&(\nu_1^*,\nu_2^*)\cdot\begin{pmatrix}1-P_{11} & -P_{21}\\ -P_{12} & 1-P_{22}\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1+P_{31}\mu_{32}\\ \lambda_2+P_{32}\mu_{32}\end{pmatrix}-z_1^*\\ \geq&(\nu_1^*,\nu_2^*)\cdot\begin{pmatrix}1-P_{11} & -P_{21}\\ -P_{12} & 1-P_{22}\end{pmatrix}^{-1}\begin{pmatrix}-P_{21}\\ 1-P_{22}\end{pmatrix}\times\mu_{22}x_{22}^*\\ =&\nu_2^*\times\mu_{22}x_{22}^*>0.\end{aligned}$$

Note that $\bar{Q}_1'(t)={}_*\bar{Q}_1'(t)$, $\bar{Q}_2'(t)={}_*\bar{Q}_2'(t)$, and ${}_*\bar{Q}_1'(t)$ and ${}_*\bar{Q}_2'(t)$ have the same sign, they must be both positive and also

$$(\nu_1^*,\nu_2^*)\cdot\begin{pmatrix}1-P_{11} & -P_{21}\\ -P_{12} & 1-P_{22}\end{pmatrix}^{-1}\begin{pmatrix}{}_*\bar{Q}_1'(t)\\ {}_*\bar{Q}_2'(t)\end{pmatrix}\geq\nu_2^*\times\mu_{22}x_{22}^*.$$

Then

$$\sum_{i=1}^3{}_*\bar{Q}_i'(t)\geq\sum_{i=1}^2{}_*\bar{Q}_i'(t)\geq C(\nu_1^*,\nu_2^*)\cdot\begin{pmatrix}1-P_{11} & -P_{21}\\ -P_{12} & 1-P_{22}\end{pmatrix}^{-1}\begin{pmatrix}{}_*\bar{Q}_1'(t)\\ {}_*\bar{Q}_2'(t)\end{pmatrix}$$

for an appropriate positive constant C. Thus, $\sum_{i=1}^3{}_*\bar{Q}_i'(t)$ has a (positive constant) lower bound.

Proof of Step 3: From (34),

$$\bar{W}'(t) = \sum_{j\in\mathcal{J}} z_j^*(1 - \sum_{i\in\mathcal{C}(j)} T_{ij}'(t)).$$

With the assumption that there are at least two nonzero $\bar{Q}_i(t)$, for each server j, $\sum_{i\in\mathcal{C}(j)} \bar{Q}_i(t) > 0$. As a result, from (35),

$$1 - \sum_{i\in\mathcal{C}(j)} T_{ij}'(t) = 0, \quad \text{for} \quad j \in \mathcal{J}.$$

Thus, $\bar{W}'(t) = 0$, which can be implied by a special case that there are at least two nonempty queues from the following lemma.

Lemma EC.1. *Assume that at a regular time t, at least two $\bar{Q}_i(t)$ are nonzero, then under any service policy such that* (35) *holds, we have*

$$\bar{W}'(t) = \sum_{i\in\mathcal{I}} y_i^* \bar{Q}_i'(t) = 0.$$

The lemma is direct and we omit the proof. From Lemma EC.1, it is enough to prove that for all $t > t_0$, there are at least two $\bar{Q}_i(t)$ being nonzero. We consider $(t_0, t_1]$ and (t_1, ∞). For (t_1, ∞), the result is obvious because $\sum_{i=1}^3 {}_*\bar{Q}_i'(t) > 0$. For $(t_0, t_1]$, note that

$$\bar{Q}_1'(t) + \bar{Q}_3'(t) = \lambda_1 + \lambda_3 + (P_{21} + P_{23})(\mu_{21} + \mu_{22}) > 0,$$

because at least one of the above terms on the right hand side should be positive.

Proof of Step 4: Assume it does not hold. Then there is one t such that $\bar{Q}(t) \neq \bar{Q}(t_2)$. Because $\bar{W}(t) = \bar{W}(t_0)$, we can conclude that $\sum_{i=1}^3 {}_*\bar{Q}_i(t) < \sum_{i=1}^3 q_i^*(\bar{W}(t_0))$. Let $\epsilon = \sum_{i=1}^3 q_i^*(\bar{W}(t_0)) - \sum_{i=1}^3 {}_*\bar{Q}_i(t)$, then due to the continuity, there is a time $s \in [t_2, t]$ such that $\sum_{i=1}^3 {}_*\bar{Q}_i(s) = \sum_{i=1}^3 q_i^*(\bar{W}(t_0)) - \frac{\epsilon}{2}$ and for $u \in (s, t)$, $\sum_{i=1}^3 {}_*\bar{Q}_i(u) < \sum_{i=1}^3 q_i^*(\bar{W}(t_0)) - \frac{\epsilon}{2}$. However, from step 2, $\sum_{i=1}^3 {}_*\bar{Q}_i'(\cdot)$ is always positive, thus it cannot decrease to $\sum_{i=1}^3 q_i^*(\bar{W}(t_0)) - \epsilon$. Hence, we arrive at a contradiction.

Now we prove (b). Assume the conclusion in (b) does not hold. Then there is a t such that $\bar{Q}(t) \neq 0$. Denote by $\epsilon = \sum_{i=1}^3 y_i^* \bar{Q}_i(t)$. Due to the continuity, there must be a time $t_1 \in (0, t)$ such that $\sum_{i=1}^3 y_i^* \bar{Q}_i(t_1) = \frac{\epsilon}{2}$. Then from step 3 above, for all $s \geq t_1$, $\sum_{i=1}^3 y_i^* \bar{Q}_i(s) = \frac{\epsilon}{2}$. Hence, we arrive at a contradiction. □

Remark EC.1. Step 3 gives a different result from that in Mandelbaum and Stolyar (2004) (see the bound using a constant $K \geq 1$ in their Theorem 3). The significance of this result is that even if the initial status is not a fixed point, the workload has no jump; therefore the proposed policy is always optimal. This is mainly due to the fact that there are no nonbasic activities according to the system structure. □

EC.3. Proof of Lemma 4

The proof is similar to the one for Lemma 6 in Chen and Ye (2012), and we provide it here for completeness. From Proposition 1, there is some time T sufficiently long, so that in any hydrodynamic limit, $\bar{Q}(t)$ will approach the fixed-point state from an initial state $\bar{Q}(0)$ with $\bar{W}(0) \le \chi + c + 1$ for $t \ge T$. Let

$$T = T_{\chi+c+1}. \tag{EC.1}$$

The initial state bound $\chi + c + 1$ is used as the subscript to remind us that T varies on the initial state.

1. The case $\ell = 0$:

Property (a): From assumption (38), $(\bar{W}^{r,0}(0), \bar{Q}^{r,0}(0)) \to (\chi, q^*(\chi))$ as $r \to \infty$. Hence, it follows from the hydrodynamic convergence (Lemma 2) and the uniform attraction (Proposition 1), that as $r \to \infty$,

$$(\bar{W}^{r,0}(u), \bar{Q}^{r,0}(u)) \to (\bar{W}(u), \bar{Q}(u)) = (\chi, q^*(\chi)), \text{ u.o.c. in } u \in [0,T]. \tag{EC.2}$$

(Because the limit is unique, the convergence is along the whole sequence of r.) Let r be sufficiently large, such that $|\bar{W}^{r,0}(u) - \chi| \le \min_{i\in\mathcal{I}} y_i^* \epsilon/2$ and $|\bar{Q}^{r,0}(u) - q^*(\chi)| \le \epsilon/2$ for all $u \in [0,T]$. Then, we have

$$\begin{aligned} |\bar{Q}^{r,0}(u) - q^*(\bar{W}^{r,0}(u))| \le & |\bar{Q}^{r,0}(u) - q^*(\chi)| + |q^*(\bar{W}^{r,0}(u)) - q^*(\chi)| \\ \le & \frac{\epsilon}{2} + |\bar{W}^{r,0}(u) - \chi| / \min_{i\in\mathcal{I}} y_i^* \le \epsilon, \end{aligned} \tag{EC.3}$$

for all $u \in [0,T]$. Hence, property (a) holds for $\ell = 0$ when r is sufficiently large.

Property (b): It follows from (EC.2) that $\bar{W}^{r,0}(u)$ is close to χ for all $u \in [0,T]$ when r is sufficiently large, which leads to property (b) for $\ell = 0$.

Property (c): From the assumption of cost function C, $q_i^*(x)$ will not be zero unless $x = 0$. Then by (EC.3), for any small enough $\epsilon_0 > 0$ with $q_i^*(\epsilon) > \epsilon_0$ and large enough r

$$\bar{Q}_i^{r,0}(u) \ge q_i^*(\bar{W}^{r,0}(u)) - \epsilon_0.$$

The increase of $q_i^*(\cdot)$ implies that $q_i^*(\bar{W}^{r,0}(u)) > q_i^*(\epsilon)$ when $\bar{W}^{r,0}(u) > \epsilon$. Thus, for any $u \in [0,T]$, $\bar{Q}_i^{r,0}(u) > 0$ for $i \in \mathcal{I}$ and

$$\bar{Y}^{r,0}(u) - \bar{Y}^{r,0}(0) = \sum_{j=1}^{2} z_j^* (u - \sum_{i\in\mathcal{C}(j)} \bar{T}_j^{r,0}(u)) = 0.$$

2. The case $\ell = 1, \cdots, \lfloor \sqrt{r}\delta/T \rfloor$: Suppose, to the contrary, there exists a subsequence $\mathcal{R}_1$ of $\{r\}$, such that for any $r \in \mathcal{R}_1$, at least one of the properties fails to hold for some integers $\ell \in [1, \sqrt{r}\delta/T]$. Then, for any $r \in \mathcal{R}_1$, there exists a smallest integer, denoted by ℓ_r, in the interval $[1, \sqrt{r}\delta/T]$, such that at least one of the properties fails to hold. To reach a contradiction, it suffices to construct an infinite subsequence $\mathcal{R}_2 \subset \mathcal{R}_1$, such that the properties hold for $\ell = \ell_r$ for sufficiently large $r \in \mathcal{R}_2$.

Property (a): From the contradictory assumption, we know that the properties hold for $\ell = 0, 1, \cdots, \ell_r - 1$, $r \in \mathcal{R}_1$. Specifically, for $\ell = \ell_r - 1$, we have $\bar{W}^{r,\ell_r-1}(0) \leq \chi + c + 1$, for all $r \in \mathcal{R}_1$. Hence, it follows from the hydrodynamic limit, that there exists a further subsequence $\mathcal{R}_2 \subset \mathcal{R}_1$, such that $(\bar{W}^{r,\ell_r-1}(u), \bar{Q}^{r,\ell_r-1}(u)) \to (\bar{W}(u), \bar{Q}(u))$, u.o.c., as $r \to \infty$ along $\mathcal{R}_2$ with $\bar{W}(0) \leq \chi + c + 1$. Then $\bar{Q}(u) = q^*(\bar{W}(u))$ for all $u \geq T$ by (EC.1). Hence, for sufficiently large $r \in \mathcal{R}_2$,

$$
\begin{aligned}
|\bar{Q}^{r,\ell_r}(u) - q^*(\bar{W}^{r,\ell_r}(u))| =& |\bar{Q}^{r,\ell_r-1}(u+T) - q^*(\bar{W}^{r,\ell_r-1}(u+T))| \\
\leq& |\bar{Q}^{r,\ell_r-1}(u+T) - \bar{Q}(u+T)| + |q^*(\bar{W}(u+T)) - q^*(\bar{W}^{r,\ell_r-1}(u+T))| \\
\leq& \epsilon,
\end{aligned}
$$

for all $u \in [0,T]$. Hence, property (a) holds with $\ell = \ell_r$ for sufficiently large $r \in \mathcal{R}_2$.

Property (c): It is similar to the proof of Property (a) in the case $\ell = 0$. From property (a) in this case above, for small enough $\epsilon_0 > 0$ with $q_i^*(\epsilon) > \epsilon_0$ and all $u \in [0,T]$, $\bar{Q}_i^{r,\ell_r}(u) \geq q_i^*(\bar{W}^{r,\ell_r}(u)) - \epsilon_0$ and then $\bar{Q}_i^{r,\ell_r}(u) > 0$ when $\bar{W}^{r,\ell_r}(u)) > \epsilon$. Thus, for any $u \in [0,T]$, $\bar{Y}^{r,\ell_r}(u) - \bar{Y}^{r,\ell_r}(0) = \sum_{j=1}^{2} z_j^*(u - \sum_{i\in\mathcal{C}(j)} \bar{T}_j^{r,\ell_r}(u)) = 0$. Hence, property (c) holds for ℓ_r with sufficiently large $r \in \mathcal{R}_2$.

Property (b): Fix any $u_0 \in [0,T]$. We consider two mutually exclusive cases: (i) The condition in (c) holds for all $\ell = 0, 1, \cdots, \ell_{r-1}$, and for $\ell = \ell_r$ with $u \leq u_0$; (ii) the condition in (c) does not hold for some $\ell \in [0, \ell_r - 1]$, or $\ell = \ell_r$ but with some $u \leq u_0$.

In the first case, $\bar{Y}^{r,\ell}(u)$ does not increase in $u \in [0,T]$, for $\ell = 0, \cdots, \ell_r - 1$ and for $\ell = \ell_r$ with $u \in [0, u_0]$. As a result, for sufficiently large r,

$$
\begin{aligned}
\bar{W}^{r,\ell_r}(u_0) =& \bar{W}^{r,0}(0) + \sum_{\ell=1}^{\ell_r-1} (\bar{W}^{r,\ell}(T) - \bar{W}^{r,\ell}(0)) + (\bar{W}^{r,\ell_r}(u_0) - \bar{W}^{r,\ell_r}(0)) \\
=& \bar{W}^{r,0}(0) + \sum_{\ell=1}^{\ell_r-1} ((y^*)^T \bar{X}^{r,\ell}(T) - (y^*)^T \bar{X}^{r,\ell}(0)) + ((y^*)^T \bar{X}^{r,\ell_r}(u_0) - (y^*)^T \bar{X}^{r,\ell_r}(0)) \\
=& \bar{W}^{r,0}(0) + (y^*)^T \bar{X}^{r,\ell_r}(u_0) - (y^*)^T \bar{X}^{r,1}(0) \\
=& \widehat{W}^r(\tau) + (y^*)^T \widehat{X}^r(\tau + \ell_r T/r + u_0/r) - (y^*)^T \widehat{X}^r(\tau + T/r) \\
\leq& (\chi + \epsilon) + [(y^*)^T \widehat{X}(\tau + \ell_r T/r + u_0/r) - (y^*)^T \widehat{X}(\tau) + \epsilon] \\
\leq& (\chi + \epsilon) + (c + \epsilon) \leq \chi + c + 1.
\end{aligned}
$$

Here, the second equality is from Property (c); the fourth equality is from (26) and (27); and the last two inequalities are from the condition of this lemma and the fact that $\widehat{X}^r \to \widehat{X}^*$ u.o.c.

In the second case, if there is a $u \in [0, u_0]$, such that $\bar{W}^{r,\ell_r}(u) \leq \epsilon$, then let $\ell_r^0 = \ell_r$ and let $u_r = \sup\{0 \leq u' \leq u_0 : \bar{W}^{r,\ell_r}(u') \leq \epsilon\}$; otherwise, let ℓ_r^0 (among $0, 1, \cdots, \ell_r - 1$) be the largest integer, such that the condition in (c) does not hold. Moreover, let $u_r = \sup\{0 \leq u' \leq T : \bar{W}^{r,\ell_r^0}(u') \leq \epsilon\}$. Then, we can conclude that $\bar{Y}^{r,\ell_r}(u)$ does not increase for $\ell_r = \ell_r^0$ and $u \geq u_r$ or $\ell_r > \ell_r^0$.

According to the definition of u_r, we can find a time point u_r^ϵ, such that

$$
u_r - \epsilon \leq u_r^\epsilon \leq u_r, \quad \text{and} \quad \bar{W}^{r,\ell_r^0}(u_r^\epsilon) \leq \epsilon.
$$

Then we have

$$\begin{aligned}
\bar{W}^{r,\ell_r}(u) =& \bar{W}^{r,\ell_r^0}(u_r^\epsilon) + \bar{W}^{r,\ell_r^0}(u_r) - \bar{W}^{r,\ell_r^0}(u_r^\epsilon) \\
& + \bar{W}^{r,\ell_r^0}(T) - \bar{W}^{r,\ell_r^0}(u_r) + \sum_{\ell=\ell_r^0+1}^{\ell_r-1} (\bar{W}^{r,\ell}(T) - \bar{W}^{r,\ell}(0)) \\
& + (\bar{W}^{r,\ell_r}(u) - \bar{W}^{r,\ell_r}(0)) \\
=& \bar{W}^{r,\ell_r^0}(u_r^\epsilon) + \bar{Y}^{r,\ell_r^0}(u_r) - \bar{Y}^{r,\ell_r^0}(u_r^\epsilon) + (y^*)^T \bar{X}^{r,\ell_r^0}(u_r) - (y^*)^T \bar{X}^{r,\ell_r^0}(u_r^\epsilon) \\
& + (y^*)^T \bar{X}^{r,\ell_r^0}(T) - (y^*)^T \bar{X}^{r,\ell_r^0}(u_r) + \sum_{\ell=\ell_r^0+1}^{\ell_r-1} ((y^*)^T \bar{X}^{r,\ell}(T) - (y^*)^T \bar{X}^{r,\ell}(0)) \\
& + ((y^*)^T \bar{X}^{r,\ell_r}(u) - (y^*)^T \bar{X}^{r,\ell_r}(0)) \\
=& \bar{W}^{r,\ell_r^0}(u_r^\epsilon) + \bar{Y}^{r,\ell_r^0}(u_r) - \bar{Y}^{r,\ell_r^0}(u_r^\epsilon) + ((y^*)^T \bar{X}^{r,\ell_r}(u) - (y^*)^T \bar{X}^{r,\ell_r^0}(u_r^\epsilon) \\
\le& \bar{W}^{r,\ell_r^0}(u_r^\epsilon) + \sum_{j\in\mathcal{J}} z_j^* \epsilon + [(y^*)^T \widehat{X}^r(\tau + \ell_r T/r + u/r) - (y^*)^T \widehat{X}^r(\tau + \ell_r^0 T/r + u_r^\epsilon/r)] \\
\le& (\chi + \epsilon) + \sum_{j\in\mathcal{J}} z_j^* \epsilon + [(y^*)^T \widehat{X}(\tau + \ell_r T/r + u/r) - (y^*)^T \widehat{X}(\tau + \ell_r^0 T/r + u_r/r) + \epsilon] \\
\le& \chi + (\sum_{j\in\mathcal{J}} z_j^* + 1)\epsilon + (c+\epsilon) \le \chi + c + 1.
\end{aligned}$$

Hence, we have shown that the properties hold for $\ell = \ell_r$ when $r \in \mathcal{R}_2$ is sufficiently large, which contradicts the definition of the subsequence $\mathcal{R}_2$. □