Optimal Interventions of Infectious Disease

Xu Sun\(^1\) and Yunan Liu\(^2\)

\(^1\)Industrial and Systems Engineering, University of Florida, USA, xusun@ufl.edu
\(^2\)Industrial and Systems Engineering, North Carolina State University, USA, yliu48@ncsu.edu

July 9, 2022

Abstract

The recent novel coronavirus outbreak provides motivation for a benefit-cost framework to guide unconventional public health interventions to reduce or stop close contact between infected and susceptible individuals. In this paper, we introduce and solve an optimal control problem in an infectious disease model where the social planner can control the transmission rate by locking down and opening the economy. The objective is to minimize the total costs, which include infection costs as well as fixed and variable costs for conducting a lockdown. We provide conditions regarding model primitives that ensure that the problem admits a simple optimal policy. Specifically, the policy specifies two switching points \((a, b)\): When the percentage of infected individuals surpasses \(b\) the social planner locks down the economy; when the percentage of infected individuals drops below \(a\), the economy is reopened. We later extend to the case where the social planner may issue multiple lockdown levels. Numerical studies are carried out to gain additional insights into the value of controls.

Keywords: infectious diseases; optimal control; benefit-cost analysis; non-pharmaceutical interventions

1 Introduction

Due to the COVID-19 pandemic, a number of non-pharmaceutical interventions, colloquially known as lockdowns, including stay-at-home orders, curfews, quarantines, and other societal
restrictions, have been implemented in numerous countries and territories around the world. These restrictions were put in place to help control the spread of the coronavirus virus. By April 2020, approximately half of the world’s population was under some form of lockdown, with governments in more than 90 countries or territories asking or ordering more than 3.9 billion people to stay at home. The scope and severity of the enforced lockdowns vary by country and territory. Several Asian countries have successfully contained COVID-19 pandemics by combining large-scale testing, contact tracing, isolation, and quarantine with moderate (e.g., South Korea) or strong (e.g., China) social-distancing measures. Recognizing that a sudden increase in COVID-19 cases could overwhelm their health-care system, many European countries, as well as the United States, followed suit by implementing aggressive social-distancing measures to combat the pandemic.

Although public health experts and economists generally supported lockdown restrictions, citing higher long-term costs for allowing the disease to spread uncontrollably, lockdown restrictions could have health, social, and economic ramifications. A lockdown-induced economic downturn can produce health problems ranging from ‘deaths of despair’ to strains on public-health budgets, potentially leading to more non-COVID-19-related deaths than a lockdown would save from the pandemic. Furthermore, social tensions can arise because of severe social confinement, which has a profound negative impact on people. As a result, a well-designed lockdown entry-and-exit strategy is critical for countries that choose to implement a lockdown policy.

Although the debate over whether lockdown measures are warranted continues, it is intuitively clear that a decision framework is required for governments and public health agencies to make well-informed decisions about the implementation of unconventional interventions against infectious diseases. In this paper, we specifically seek preliminary answers to the following policy questions: (i) Is a lockdown necessary? (ii) If a lockdown is deemed economically beneficial, when is the best time to implement it and when is it best to remove it?

To that end, and to gain analytical insights into the effects of various socioeconomic factors on government intervention strategies, we introduce a stochastic control formulation comprised of a stylized disease-spread model and a specific type of governmental intervention. The disease-spread model captures three realistic aspects of how an infectious disease can spread among a given population: (i) the disease can be passed from one infected individual to another who is
virus-free through random contact; (ii) an infected individual may recover from the disease after a certain amount of time; and (iii) in addition to contact-based infection and self-healing, other unknown environmental factors can contribute to disease spread, implying that the trajectory of disease-spread is not completely predictable. The type of governmental intervention under consideration has the following binary structure: the social planner can choose when to lock down and open the economy at any point in time. In particular, when a lockdown is activated, contact is reduced, and the transmission rate is reduced to a lower level. Meanwhile, locking down imposes socioeconomic costs. At a high level, the problem can be viewed as an optimal switching problem with the goal of optimally balancing the burden induced by the disease with the economic cost of lockdowns.

Although there is a large body of literature on infectious disease control, the majority of it is based on a deterministic disease-spread model (thus ignoring the inherent randomness in disease transmission) and focuses on pharmaceutical interventions such as vaccination. Instead, we consider randomness in the disease-spread process and focus on unconventional government interventions, which were widely observed during the COVID-19 pandemic. In this regard, our paper contributes to the literature on epidemic control. We do recognize the complexity of the disease spread process and more accurate infectious disease models may need to include several more compartments (e.g., see [El Housni et al. (2022a)]), but we emphasize that the present paper aims to obtain tractable insights using a relatively simple model.

Our main findings are summarized as follows: Implementing a lockdown policy is never optimal when the economic cost of a lockdown policy is prohibitively high compared to the long-term cost of allowing the disease to spread uncontrollably. When a lockdown is not prohibitively expensive, the social planner should employ the following strategy: When the number of infected individuals exceeds a certain threshold $b$, it is best to implement a lockdown and then reopen the economy when the number falls below another threshold $a < b$. This structure is intuitive, yet determining the switching boundaries $a$ and $b$ necessitates a formal analysis, which we do in this paper. And we also extend our framework to allow for multiple lockdown levels; under certain conditions, we show that a similar multi-level sequential switching policy is optimal.

The rest of the paper is structured as follows. Section 2 reviews the relevant literature. Section 3 introduces our disease-spread model and presents the stochastic control problem. Section 4 characterizes the optimal intervention strategy via the solution to the corresponding
dynamic programming equation; in the same section, we conduct numerical explorations to gain additional insights into the structure of the optimal policy. Section 5 discusses a useful extension of the base model that considers multiple lockdown levels; we show that, under certain conditions, the optimal policy is a multi-level sequential switching rule, and we provide a procedure to compute all switching thresholds. Section 6 concludes by highlighting some limitations and suggesting future research directions.

2 Literature Review

The paper builds on the literature on stochastic models of infectious diseases; see, e.g., Mao et al. (2002), Tuckwell and Williams (2007), Gray et al. (2011), Cai et al. (2017). This line of research was mostly concerned with the persistence and extinction of diseases. By contrast, we focus on control mechanisms for an infectious disease and attempt to inform decisions about the best timing of government intervention.

The present work is related to several recent studies that have established analytical results for the optimal control of an infectious disease using deterministic infectious disease models. Sethi and Staats (1978) used optimal control to minimize the cost of both disease and medical treatment. Behncke (2000) studied a general type of susceptible-infected-recovered model with control by vaccination, quarantine, or health campaigns; the author showed that in all cases, it is optimal to exert maximum effort. Chehrazi et al. (2019) investigated an optimal control problem for an susceptible-infected-susceptible (SIS) model of an infectious disease with resistance and proved that the optimal prescription policy is of the bang-bang type with a single switching time. Chen and Kong (2022) investigated how different hospital admission policies affect the spread of an infectious disease by considering a modified susceptible-exposed-infected-recovered (SEIR) model under a static social distancing policy. El Housni et al. (2022a,b) developed new compartmentalized models to study the effects of testing capacity and social distancing measures on the evolution of the pandemic in New York City and North Carolina. A recent work by Chen et al. (2022) developed a deterministic susceptible-infected-recovered (SIR) model under a finite number of test kits and studied the impact of the management of the test capacity on the disease transmission process. By contrast, our paper adopts a stochastic modeling framework. Besides offering an extra degree of realism—by capturing the stochastic nature of disease transmission—the stochastic model considered in this paper avoids the need to assume an artificial threshold,
as was often done in deterministic models to capture eradication/extinction; see, e.g., Tebbens and Thompson (2009).

Our literature search reveals a limited number of papers on the control of stochastic epidemic models. Le Févére (1981) derived the optimal control policy for a birth-and-death epidemic model assuming both birth and death rates (representing quarantine and medical treatment respectively) are subject to control; in addition, the author examined the effect of model parameters on the optimal strategy. Focusing on vaccination and transmission reduction measures, Yaesoubi and Cohen (2011) formulated a Markov decision process to derive optimal vaccination and transmission-reducing intervention. The recent work by Lee et al. (2022) conducted comprehensive agent-based simulations to study how the COVID-19 pandemic responds to several non-pharmaceutical interventions. Our work is most closely related to Bai et al. (2021) which studied an optimal public health intervention problem with the goal of balancing the cost of infection and socioeconomic losses. Bai et al. (2021) modeled disease dynamics by a deterministic SIR process, and a major consideration of their work is the response from strategic individuals each of which aims to maximize its own utility. Unlike these papers that considered rate control driven by vaccination and/or treatment, we formulate an optimal switching problem, motivated by the mass implementation of non-pharmaceutical interventions across the globe during the COVID-19 crisis. Our work is further differentiated because we study Brownian models of system evolution which enable us to derive explicit solutions and clear-cut insights.

There has been a stream of literature that studies stochastic control problems involving sequential switching decisions. In a nutshell, these problems involve switching costs that can be considered fixed investments required to realize advantages of an appropriate regime. Intuitively, such costs force the controller to look beyond the immediate advantages to ensure that a regime switch will accrue sufficient benefit over time to merit the fixed investment. The problems are formulated either using a discounted cost criterion (Duckworth and Zervos 2001, Ly Vath and Pham 2007, Pham et al. 2009, Zervos et al. 2013) or an average cost criterion (Wu and Chao 2014, Matoglu et al. 2019, Vande Vate 2021). The contributions of these authors are centered on characterization of solutions to the corresponding Bellman equation and assume that the underlying state process is well-behaved in the sense that it fulfills the standard Lipschitz continuity condition (i.e., the drift and volatility are Lipschitz continuous functions of the state process). Our state process does not satisfy the desired Lipschitz continuity assumption due
to the fact that the disease spread shows super-linear growth. Indeed, unlike the majority of papers in this line of work, our dynamic programming equation features a non-Lipschitz singularity at the boundary, posing non-trivial technical challenges for our analysis. Thus, our paper contributes to the literature on optimal switching by developing a set of tools that we believe carry methodological novelty.

In brief, the present paper seems to be the first to study the effect of non-pharmaceutical interventions against a pandemic under a formal stochastic control framework, taking into account the inherently random nature of epidemic growth and spread.

3 Model

Notation and Convention We assume that all random quantities are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Expectation with respect to $\mathbb{P}$ is $\mathbb{E}$. For a sufficiently smooth function $f$, we use $f'$ and $f''$ to denote its first and second derivatives. The partial derivative of a multivariate function $f(x, y, \ldots)$ with respective to variable $x$ is denoted by $f_x$.

Consider a social planner who is evaluating interventions for an infectious disease affecting a population through a type of binary decisions, where the two decisions are “lock down the economy” and “open the economy”. To formulate the social planner’s problem, we start by establishing a mathematical framework of disease spread in the absence of any intervention in §3.1. Our base formulation extends the classical SIS epidemic model from a deterministic framework to a stochastic one where the number of infectious individuals follows a stochastic differential equation (SDE). Then in §3.2 we introduce the intervention mechanism, present relevant socioeconomic costs, and state the social planner’s problem.

3.1 Disease Dynamics

In a closed population of fixed size, there are two types of individuals: susceptible individuals and infected individuals. We make the simplifying assumption that all infected individuals exhibit symptoms and will report their change of status (from susceptible to infected) upon contracting the disease.

The percentage of infected people, denoted by $X_t$, describes the system state at any point in time, $t \geq 0$. Thus, for all $t \geq 0$, $X_t$ is in the range $[0, 1]$, and the percentage of the sus-
ceptible population at time $t$ is given as $1 - X_t$. An infected individual passes the disease to a susceptible one via contact at some rate determined by both the host-population density and the infectiousness of the pathogens. Moreover, all infected individuals are able to recover at a rate $\gamma$. Here, recovery means that the virus is eliminated from the body, so the parameter $\gamma$ reflects the strength of the host’s immune system. We assume that all individuals who recover return to the susceptible state, which effectively restricts our analysis to infections with negligible mortality rates and no conferred immunity. This is an applicable assumption for viruses that can continually mutate, enabling the virus to evade an individual’s immune system. In the absence of any intervention, the dynamics can be described by the following SDE:

$$dX_t = \left( \beta(1 - X_t) - \gamma \right) X_t dt + \sigma \sqrt{X_t(1 - X_t)} dB_t,$$

where the first drift term, $\beta X_t(1 - X_t)$, represents the disease spread dynamics, the second drift term, $\gamma X_t$, models the natural recovery dynamics, and the third term, $B_t$ denotes a standard Brownian motion. Here $\beta$ can be interpreted as the rate of infection due to all activities (without lockdown), including both essential activities (e.g., grocery shopping and doctor visits) and nonessential social interactions (e.g., social gatherings and entertainment events). The parameter $\gamma$ is the recovery rate of an infected individual, and $\sigma$ is the volatility parameter that characterizes the magnitude of the stochastic fluctuations of the system.

In above equation, direct verification of conditions of Theorem 5 in Feller (1954) reveals that zero is an accessible and absorbing state, meaning that the process will hit the zero boundary (without intervention) in a finite (yet random) time and will stay at zero thereafter. It is also noteworthy that our disease model belongs to the class of Wright-Fisher processes, see, e.g., Eq. (1) in Jenkins and Spano (2017).

Model (1) implicitly assumes that the disease’s incubation period is zero. Moreover, by assuming that the rate parameter $\beta$ is constant, we have ignored potential behavior changes in response to the epidemic’s growth: it is well known that the use of static parameters can lead to overestimation of disease incidence or prevalence. Nonetheless, we routinely make these simplifying assumptions in order to achieve greater mathematical tractability.

### 3.2 Intervention Mechanism and the Social Planner’s Problem

We consider the possibility that the social planner can activate or deactivate a lockdown which curtails the transmission rate upon enactment. With a lockdown in place, the disease is trans-
mitted at a (potentially much) slower rate $\tilde{\beta} < \beta$. In contrast to $\beta$ that characterizes the rate of infection due to both essential and nonessential activities, $\tilde{\beta}$ measures the rate of infection due to only essential activities to maintain a living (e.g., grocery shopping and doctor visits).

To perform a formal cost-benefit analysis, define $\ell > 0$ as the marginal cost of infection, so the cost of infection will accumulate at the rate $\ell X_t$. This cost may include symptom relief, hospitalizations, long-term infection complications, and lost productivity. We consider the costs of locking down the economy in addition to the costs of infections. We assume, in particular, that each time the social planner enacts a lockdown, it incurs a fixed cost of $K$; additionally, a lockdown incurs economic tolls at the rate of $\kappa$ for the duration of its implementation.

Because the epidemic may die out even without intervention, it is intuitively clear that locking down the economy is an appealing option only when the cost of implementing them is not prohibitively expensive. At any given point in time, the system can reside in one of the two modes: “on” and “off”. The system is said to be in its on mode if a lockdown policy is in effect and in its off mode if the policy is not in effect. The transition from one mode to the other is immediate and forms a sequence of decisions made by the social planner. These recurring decisions constitute an intervention strategy, which we model using an adapted, finite variation, càdlàg process $Y$ with values in the set $\{0, 1\}$. Specifically, $Y_t = 0$ means that the system is in the off mode, whereas $Y_t = 1$ means that the system is in the on mode. Switching from the off to the on mode triggers a fixed cost $K$, whereas switching from the on to the off mode is free. The objective of the social planner is to minimize the total cost. More formally, the social planner aims to identify some $Y$ to

$$
\text{minimize} \quad J(x, y) := \mathbb{E} \left[ \int_0^\tau \ell X_u du + \kappa \int_0^\tau Y_u du + K \sum_{u \leq \tau} [\Delta Y_u]^+ \left| X_0 = x, Y_0 = y \right. \right],
$$

where the stopping time $\tau$ denotes the time at which $X_t$ hits zero for the first time, and $X_t$ is the solution to the SDE

$$
dX_t = \left[ b(Y_t)(1 - X_t) - \gamma \right] X_t dt + \sigma \sqrt{X_t(1 - X_t)} dB_t
$$

for $b(y) := \beta 1_{\{y=0\}} + \tilde{\beta} 1_{\{y=1\}}$. See Table 1 for a summary of all notation.

In principle, the optimal strategy can be an arbitrary functional of the sample paths of $X$. However, due to the strong Markov property of the process $X$, one can restrict attention to the class of Markov policies.
4 Analysis

This section is devoted to solving the control problem posited in the previous section. In §4.1, we describe the dynamic programming equation that will be used to characterize the optimal strategy. In §4.2, we characterize the optimal intervention strategy via the solution to the dynamic programming equation.

4.1 Dynamic Programming Equation

Let $V(x, y)$ denote the value function associated with the problem (2). With reference to the general control theory, the function $V(x, y)$ ought to solve the following dynamic programming equation in the form of a variational inequality (VI):

$$
\min \left\{ \frac{\sigma^2}{2} x(1-x) V_{xx}(x, y) + \left( b(y)x - b(y)x^2 - \gamma x \right) V_x(x, y) \\
+ \ell x + \kappa y, \ V(x, 1) + K - V(x, 0), \ V(x, 0) - V(x, 1) \right\} = 0,
$$

subject to the condition $V(0, y) = 0$ and the requirement that $\lim_{x \to 1} V_x(x, y)$ exists and is finite. In general, a VI formulation reveals structural properties of the optimal policy by reducing a dynamic decision problem to a point-wise optimization problem.

To gain some insight into the origin of the dynamic programming equation, suppose that...
the economy is locked down at time 0, i.e., \( Y_0 = 1 \). In light of the problem setup, the social planner’s immediate decisions consist of choosing between two options. The first is to continue enforcing the lockdown policy for a short time, \( \Delta t \), and then continue optimally. With regard to Bellman’s principle of optimality, we must have

\[
V(x, 1) \leq \mathbb{E} \left[ \int_0^{\Delta t} \ell_X \, du + \kappa \int_0^{\Delta t} Y_u \, du + V(X_{\Delta t}, 1) \right],
\]

where the inequality is due to the fact that continuing locking down the economy for the next moment may not be optimal. Assuming that \( V(x, 1) \) is sufficiently smooth, we can apply Itô’s lemma to the second term, divide both sides by \( \Delta t \), and send \( \Delta t \to 0 \) to get

\[
\frac{\sigma^2}{2} x(1-x)V_{xx}(x, 1) + \left( \beta x - \beta x^2 - \gamma x \right) V_x(x, 1) + \ell x + \kappa \geq 0.
\]

The second option is to switch to the off mode and then continue optimally, resulting to

\[
V(x, 1) \leq V(x, 0).
\]

Since these two are the social planner’s only options, we can conclude that one of the two preceding inequalities must hold as an equality. Thus, the function \( V(x, 1) \) must satisfy

\[
\min \left\{ \frac{\sigma^2}{2} x(1-x)V_{xx}(x, 1) + \left( \beta x - \beta x^2 - \gamma x \right) V_x(x, 1) + \ell x + \kappa, \ V(x, 0) - V(x, 1) \right\} = 0. \tag{5}
\]

Arguing along similar lines, we can conclude that the function \( V(x, 0) \) must satisfy

\[
\min \left\{ \frac{\sigma^2}{2} x(1-x)V_{xx}(x, 0) + \left( \beta x - \beta x^2 - \gamma x \right) V_x(x, 0) + \ell x, \ V(x, 1) + K - V(x, 0) \right\} = 0. \tag{6}
\]

Combining (5) and (6) leads to the desired VI (4).

We now postulate the solution structure of the dynamic programming equation. Intuitively, if the costs associated with a lockdown are not prohibitively high, it would be beneficial for the social planner to implement a lockdown policy on a temporary basis. This intuition leads us to conjecture that the optimal control strategy is a sequential switching policy comprised of the following actions. If the system is currently operating in its off mode, then it is optimal to remain in that mode if \( X \) is below a threshold, say \( x^*_1 \), and switch to its on mode once \( X \) rises above \( x^*_1 \). On the other hand, if the system is currently operating in its on mode, then it is optimal to remain in that mode if \( X \) is above a certain level, say \( x^*_0 \), and switch to its off mode as soon as \( X \) drops below \( x^*_0 \). Clearly, this strategy is well-defined if \( x^*_0 < x^*_1 \). Moreover, if this
strategy, henceforth denoted as $Y^*$, is indeed optimal, we should be able to find $V_0(\cdot)$ and $V_1(\cdot)$ such that

\[
\frac{\sigma^2}{2} x (1-x)V''_0(x) + (\beta x - \beta x^2 - \gamma x) V'_0(x) + \ell x = 0 \quad \text{for} \quad x \in [0, x^*_1) \quad \text{and} \quad (7)
\]

\[
\frac{\sigma^2}{2} x (1-x)V''_1(x) + (\tilde{\beta} x - \tilde{\beta} x^2 - \gamma x) V'_1(x) + \ell x + \kappa = 0 \quad \text{for} \quad x \in (x^*_0, 1] \quad (8)
\]

subject to the boundary conditions:

\[
V_0(x) = V_1(x) \quad \text{for} \quad x \in [0, x^*_0] \quad \text{and} \quad V_0(x) = V_1(x) + K \quad \text{for} \quad x \in [x^*_1, 1], \quad (9)
\]

plus two optimality conditions derived from the “principle of smooth fit”:

\[
V'_0(x^*_0) = V'_1(x^*_0) \quad \text{and} \quad V'_0(x^*_1) = V'_1(x^*_1). \quad (10)
\]

### 4.2 Characterizing the Optimal Policy

Note that Equation (7) does not involve the unknown function $V$ itself. Therefore, it is essentially a first-order differential equation. Indeed, if letting $U_0 := V'_0$, then $U_0$ solves

\[
U'_0(x) + \frac{2}{\sigma^2} \left( \beta - \frac{\gamma}{1-x} \right) U_0(x) = -\frac{2}{\sigma^2} \frac{\ell}{1-x}. \quad (11)
\]

To proceed, let

\[
\phi(x, \iota) := e^{-2\beta x/\sigma^2} (1-x)^{-2\gamma/\sigma^2} \left( \iota - \frac{2\ell}{\sigma^2} \int_0^x e^{2\beta u/\sigma^2} (1-u)^{2\gamma/\sigma^2-1} du \right). \quad (12)
\]

It is straightforward to verify that $\phi(\cdot, \iota)$ is a solution to (11) for each fixed $\iota$, and $\phi(0, \iota) = \iota$.

**Proposition 1** There exists a unique $\overline{\iota}$ such that $\lim_{x \to 1} \phi(x, \overline{\iota})$ exists and is finite.

**Proof of Proposition 1.** Let

\[
\overline{\iota} := \frac{2\ell}{\sigma^2} \int_0^1 e^{2\beta u/\sigma^2} (1-u)^{2\gamma/\sigma^2-1} du. \quad (13)
\]

Substituting (13) into (12) yields

\[
\phi(x, \overline{\iota}) = e^{-2\beta x/\sigma^2} (1-x)^{-2\gamma/\sigma^2} \frac{2\ell}{\sigma^2} \int_x^1 e^{2\beta u/\sigma^2} (1-u)^{2\gamma/\sigma^2-1} du.
\]

By applying l'Hospital rule, we obtain

\[
\lim_{x \to 1} \phi(x, \overline{\iota}) = \lim_{x \to 1} \frac{-2\ell}{\sigma^2} e^{2\beta x/\sigma^2} (1-x)^{2\gamma/\sigma^2-1} - \frac{2\ell}{\sigma^2} e^{2\beta x/\sigma^2} (1-x)^{2\gamma/\sigma^2-1} = \ell / \gamma.
\]
It is easy to see that \( \lim_{\ell \to 1^-} \phi(x, \ell) = \infty \) for \( \ell > \bar{\ell} \) and \( \lim_{\ell \to 1^+} \phi(x, \ell) = -\infty \) for \( \ell < \bar{\ell} \), meaning that \( \ell = \bar{\ell} \) is the only initial value for which \( \lim \phi(x, \ell) \) is finite. The proof is thus complete. \( \square \)

Similarly, by letting \( U_1 := V_1' \), from (8) we can see that \( U_1 \) solves the following differential equation:

\[
U_1'(x) + \frac{2}{\sigma^2} \left( \beta - \frac{\gamma}{1-x} \right) U_1(x) = -\frac{2}{\sigma^2} \left( \frac{\ell}{1-x} + \frac{\kappa}{x(1-x)} \right).
\]  

Consider the class of functions \( \{ \psi(\cdot, c) \} \) indexed by \( c \), where each \( \psi(\cdot, c) \) is defined as

\[
\psi(x, c) := e^{2\tilde{\beta}(1-x)/\sigma^2} (1-x)^{-2\gamma/\sigma^2} \left[ \frac{2}{\sigma^2} \int_0^{1-x} e^{-2\tilde{\beta}u/\sigma^2} u^{2\gamma/\sigma^2 - 1} \left( \ell + \frac{\kappa}{1-u} \right) du + c \right].
\]

It is straightforward to check that \( \psi(\cdot, c) \) is a solution to (14) for each fixed \( c \). It is also easy to verify that \( \lim_{x \to 1^-} \psi(x, c) \) exits and is finite if and only if \( c = 0 \). And \( \lim_{x \to 1^-} \psi(x, c) = (\ell + \kappa)/\gamma \). For simplicity, in what follows we will simply write \( \psi(x, c) \) as \( \psi(x) \) when \( c = 0 \). Thus,

\[
\psi(x) = e^{2\tilde{\beta}(1-x)/\sigma^2} (1-x)^{-2\gamma/\sigma^2} \frac{2}{\sigma^2} \int_0^{1-x} e^{-2\tilde{\beta}u/\sigma^2} u^{2\gamma/\sigma^2 - 1} \left( \ell + \frac{\kappa}{1-u} \right) du.
\]  

The next result, which examines the number of intersections that \( \phi(\cdot, \ell) \) and \( \psi(\cdot) \) can have, is a key stepping stone towards the main result to be stated in Theorem 1.

**Lemma 1** For any \( \ell \leq \bar{\ell} \), \( \phi(\cdot, \ell) \) and \( \psi(\cdot) \) can intersect at most twice on the open interval \((0, 1)\).

**Proof of Lemma 1.** Define \( \varphi(\cdot, \ell) := \phi(\cdot, \ell) - \psi(\cdot) \). Suppose, by way of contradiction, that \( \phi(\cdot, \ell) \) and \( \psi(\cdot) \) cross more than twice. Then for each fixed \( \ell \), the function graph of \( \varphi(\cdot, \ell) \) must have crossed the horizontal line at least four times, two times from below and two times from above. On the other hand, since \( \phi \) and \( \psi \) are solutions to Equations (11) and (14), we can use the two equations to deduce that

\[
\frac{\sigma^2}{2} x(1-x) \varphi'(x, \ell) = \kappa - x(1-x)\psi(x) \quad \text{whenever} \quad \varphi(x, \ell) = 0.
\]  

A tedious yet straightforward calculation involving (15) will show that \( x(1-x)\psi(x) \) is strictly quasi concave. This, however, implies that the function graph of \( \varphi \) can the horizontal line at most three times thanks to (16), leading to a contradiction. Therefore, \( \phi(\cdot, \ell) \) and \( \psi(\cdot) \) can cross at most twice, as desired. \( \square \)

**Assumption 1** The two functions, \( \phi(\cdot, \bar{\ell}) \) and \( \psi(\cdot) \), intersect at two points, \( \bar{x}_0 \) and \( \bar{x}_1 \), and

\[
K \leq \bar{K} := \int_{\bar{x}_0}^{\bar{x}_1} (\phi(y, \bar{\ell}) - \psi(y)) dy.
\]  

12
Theorem 1 (Sequential switching policy) Suppose that Assumption 1 is satisfied. (i) There exists some \( \tau^* \leq \bar{\tau} \) such that \( \phi(\cdot, \tau^*) \) and \( \psi(\cdot) \) intersect exactly twice at some \( x_0^* \) and \( x_1^* \), such that
\[
K = \int_{x_0^*}^{x_1^*} (\phi(y, \tau^*) - \psi(y)) \, dy.
\]
(ii) A sequential switching policy characterized by \( (x_0^*, x_1^*) \) is optimal for Problem (2).

Remark 1 There are two possible ways Assumption 1 can be violated. It’s possible that the function graphs of \( \phi(\cdot, \tau^*) \) and \( \psi(\cdot) \) never intersect. They could also intersect, but the area of intersection, \( K \) is less than the fixed cost \( K \). In both cases, we interpret the violation to mean that intervention is too costly and therefore not worth considering; that is, it is optimal never to lock down.

Proof of Theorem 1 To establish part (i), we know from Proposition 1 that \( \phi(x, \tau) \) decreases to negative infinity for any fixed \( x \in [0, 1] \) as \( \tau \) approaches negative infinity. Therefore, there exists some \( \underline{\tau} \in (-\infty, \bar{\tau}) \) such that \( \phi(x, \tau) \) and \( \psi(\cdot) \) do not intersect for any \( \tau < \underline{\tau} \), intersect but do not cross (i.e., touch each other) for \( \tau = \underline{\tau} \), and intersect at least twice for any \( \tau \in (\underline{\tau}, \bar{\tau}] \). On the other hand, part (i) of the lemma reveals that the number of points of intersection that \( \phi(\cdot, \tau) \) and \( \psi(\cdot) \) have can be at most two. It follows that \( \phi(x, \tau) \) and \( \psi(\cdot) \) must intersect exactly twice for \( \tau \in (\underline{\tau}, \bar{\tau}] \). For such \( \tau \), let \( x_0(\tau) \) and \( x_1(\tau) \) denote the corresponding two points at which \( \phi(\cdot, \tau) \) and \( \psi(\cdot) \) intersect. Then
\[
x_1(\tau) - x_0(\tau) \to 0 \quad \text{and} \quad \int_{x_0(\tau)}^{x_1(\tau)} [\phi(x, \tau) - \psi(x)] \, dx \to 0 \quad \text{as} \quad \tau \to \underline{\tau}.
\]
When \( \tau = \bar{\tau} \), \( x_0(\bar{\tau}) = \bar{x}_0 \) and \( x_1(\bar{\tau}) = \bar{x}_1 \). Since \( x_0(\tau) \) and \( \phi(x, \tau) \) are monotonically increasing in \( \tau \), and \( x_1(\tau) \) is monotonically decreasing in \( \tau \), \( \int_{x_0(\tau)}^{x_1(\tau)} [\phi(x, \tau) - \psi(x)] \, dx \) increases from 0 to \( K \), which is greater than or equal to \( K \) by our hypothesis. Hence, we conclude that there exists some \( \tau^* \in (\underline{\tau}, \bar{\tau}] \) such that
\[
\int_{x_0(\tau^*)}^{x_1(\tau^*)} [\phi(x, \tau^*) - \psi(x)] \, dx = K.
\]
Denoting \( x_0^* = x_0(\tau^*) \) and \( x_1^* = x_1(\tau^*) \), we complete the proof of part (i).

Towards establishing part (ii), let \( V(x, y) \) be such that (a) \( V(0, 0) = 0 \), (b) \( V(x_0^*, 0) = V(x_0^*, 1) \), (c) \( V_x(x, 0) = \phi(x, \tau^*)1_{\{x < x_1^*\}} + \psi(x)1_{\{x \geq x_1^*\}} \) and \( V_x(x, 1) = \phi(x, \tau^*)1_{\{x \leq x_0^*\}} + \psi(x)1_{\{x > x_0^*\}} \).

By construction, \( V(x, y) \) satisfies the variational inequality (4). For each \( n \geq 1 \), define \( \tau_n := \tau \land n \),
where we recall that \( \tau \) is the random time when \( X \) hits zero for the first time. Using the Itô-Tanaka formula, we can find that

\[
V(X_t, Y_t) = V(x, y) + \int_0^t \left[ \frac{\sigma^2}{2} X_u (1 - X_u) V_{xx}(X_u, Y_u) + (b(Y_u) (1 - X_u) - \gamma) X_u V_x(X_u, Y_u) \right] du \\
+ \int_0^t \sqrt{X_u (1 - X_u)} V_x(X_u, Y_u) dB_u + \sum_{u \leq t} [V(X_u, Y_{u^+}) - V(X_u, Y_u)].
\]

This implies that

\[
\int_0^t \ell X_u du + \kappa \int_0^t Y_u du + K \sum_{u \leq t} [\Delta Y_u]^+ = V(x, y) - V(X_t, Y_t) + \int_0^t \left[ \frac{\sigma^2}{2} X_u (1 - X_u) V_{xx}(X_u, Y_u) \\
+ (b(Y_u) (1 - X_u) - \gamma) X_u V_x(X_u, Y_u) + \ell X_u + \kappa Y_u \right] du \\
+ \int_0^t \sqrt{X_u (1 - X_u)} V_x(X_u, Y_u) dB_u \\
+ \sum_{u \leq t} [V(X_u, Y_{u^+}) - V(X_u, Y_u) + K] [\Delta Y_u]^+ + \sum_{u \leq t} [V(X_u, Y_{u^+}) - V(X_u, Y_u)] [\Delta Y_u]^-. 
\]

Because \( V(x, y) \) satisfies (4), we know that

\[
\int_0^t \ell X_u du + \kappa \int_0^t Y_u du + K \sum_{u \leq t} [\Delta Y_u]^+ \geq V(x, y) - V(X_t, Y_t) + \int_0^t \sqrt{X_u (1 - X_u)} V_x(X_u, Y_u) dB_u.
\]

Letting \( t = \tau_n \), taking expectations on both sides, and noting that \( \tau_n \) is bounded, we obtain

\[
\mathbb{E} \left[ \int_0^{\tau_n} \ell X_u du + \kappa \int_0^{\tau_n} Y_u du + K \sum_{u \leq \tau_n} [\Delta Y_u]^+ \right] \geq V(x, y) - \mathbb{E} [V(X_{\tau_n}, Y_{\tau_n})]. \tag{19}
\]

Since \( \tau_n \to \tau \) almost surely as \( n \to \infty \), we can apply the monotone convergence theorem to conclude

\[
\mathbb{E} \left[ \int_0^{\tau_n} \ell X_u du + \kappa \int_0^{\tau_n} Y_u du + K \sum_{u \leq \tau_n} [\Delta Y_u]^+ \right] \to \mathbb{E} \left[ \int_0^\tau \ell X_u du + \kappa \int_0^\tau Y_u du + K \sum_{u \leq \tau} [\Delta Y_u]^+ \right].
\]

Similarly, we can apply the bounded convergence theorem to obtain

\[
\mathbb{E} [V(X_{\tau_n}, Y_{\tau_n})] \to \mathbb{E} [V(X, Y_\tau)] = 0,
\]

where the equality uses the fact that \( X_\tau = 0 \) and the boundary condition that \( V(0, 0) = 0 \).

Thus, by passing to the limit \( n \to \infty \) in (19), we obtain

\[
V(x, y) \leq \mathbb{E} \left[ \int_0^\tau \ell X_u du + \kappa \int_0^\tau Y_u du + K \sum_{u \leq \tau} [\Delta Y_u]^+ \right]. \tag{20}
\]
If $Y^*$ is the sequential switching policy with switching boundaries $x^*_0$ and $x^*_1$, then (18) holds with equality. By adopting the same argument as before, we can conclude that (20) holds with equality, which completes the proof of part (ii).

We next develop some numerical examples to illustrate our main results. Let $\beta = 1, \bar{\beta} = 0.2, \gamma = 1, \sigma = 0.5, \ell = 1, \kappa = 0.2$, and $K = 0.2$. Following Lemma 1, we compute $\bar{K} = 0.266$, $\bar{\iota} = 3.92, \iota^* = 3.86$, $x^*_0 = 0.033$, and $x^*_1 = 0.493$. In Figure 1, we graph three functions $\psi(x)$, $\phi(x, \bar{\iota})$ and $\phi(x, \iota^*)$ as functions of $x$. Here the area of the shaded region is $K = 0.2$.

![Figure 1: Illustration of the optimal switching policy: $x^*_0 = 0.033, x^*_1 = 0.493$, with $\beta = 1, \bar{\beta} = 0.2, \gamma = 1, \sigma = 0.5, \ell = 1, \kappa = 0.2, K = 0.2, \bar{K} = 0.266, \bar{\iota} = 3.92, \iota^* = 3.86$.](image)

To visualize the dynamics of state process under the optimal intervention policy, we provide simulation results of a sample path of $X_t$ and $Y_t$; see Figure 2. We can see that the state process is somewhat controlled within a band $(x^*_0, x^*_1)$ when $Y_t$ follows our sequential switch policy.
Figure 2: Sample path of the state process and control process under the optimal policy with: $x_0 = 0.033$, $x_1 = 0.493$, $\beta = 1$, $\beta = 0.2$, $\gamma = 1$, $\sigma = 0.5$.

5 Extension to Multiple Lockdown Levels

To this point, we have considered situations where the social planner can only put the system in one of the two modes. In practice, there may be multiple modes due to different levels of lockdown (e.g., a full vs. a partial lockdown) being available. For example, the novel “hybrid” teaching format at many universities which allows half of the students to appear in the classroom (with the other other attending online) can be interpreted as a one-half lockdown.

Motivated by this observation, we are now considering scenarios where the social planner can implement different levels of lockdown and make policy recommendations on how to best use lockdown measures. Formally, we assume that there are $m$ lockdown levels, indexed by $i = 1, \ldots, m$, giving rise to $m + 1$ different modes. Under level $i$ lockdown, the transmission rate will be reduced from $\beta$ to $\beta_i$, but economic tolls will be incurred continuously at the rate $\kappa_i$. We assume

$$\beta > \beta_1 > \cdots > \beta_m \quad \text{and} \quad 0 < \kappa_1 < \cdots < \kappa_m,$$

so that the lockdown measure becomes more stringent and more costly as the index of the
lockdown level increases.

We model the social planner’s decisions as an adapted, finite variation, càdlàg process $Y$, but with values in the set $\{0, 1, \ldots, m\}$. Specifically, $Y_t = 0$ indicates that nothing is being done, whereas $Y_t = i$ indicates that level $i$ lockdown is in effect. For the ease of discussion, we also refer to doing nothing as the level 0 lockdown. We assume that switching is only possible between adjacent levels and that upgrading from level $i$ to level $i + 1$ lockdown incurs a fixed cost $K_{i, i+1}$, whereas downgrading from level $i + 1$ lockdown is free. This assumption is consistent with reality because it is often easier for the public to accept social changes if the changes are small and gradual. It should be noted that removing this assumption (i.e., allowing a switch-over to occur between arbitrary pairs) can give rise to enormous technical difficulties. Indeed, as observed by Chernoff and Petkau (1978), for problems involving more than two control regimes, “the analytic approach becomes cumbersome.”

Let $k(y) = \sum_{i=1}^{m} \kappa_i 1_{\{y = i\}}$ and define

$$K(u, v) := \begin{cases} K_{u,v} & \text{if } (u, v) = (i, i + 1) \text{ for } i = 0, \ldots, m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

The objective of the social planner now becomes to

$$\min_{\tau} \mathbb{E} \left[ \int_0^\tau \ell X_u du + \int_0^\tau k(Y_u) du + \sum_{u \leq \tau} K(Y_u, \Delta Y_u) X_u \right] \bigg| X_0 = x, Y_0 = y ;$$

in the above, the stopping time $\tau$ again denotes the time at which $X$ hits zero for the first time, while $X$ now satisfies the SDE

$$dX_t = [b(Y_t)(1 - X_t) - \gamma] X_t dt + \sigma \sqrt{X_t(1 - X_t)} dB_t,$$

where $b$ is redefined as $b(y) = \beta 1_{\{y = 0\}} + \sum_{i=1}^{m} \beta_i 1_{\{y = i\}}$.

As before, let $V(x, y)$ denote the value function associated with the problem. With reference to the general control theory, we anticipate the function $V$ to satisfy

$$\min \left\{ \frac{\sigma^2}{2} x (1 - x) V_{xx}(x, y) + (b(y)x - b(y)x^2 - \gamma x) V_x(x, y) + \ell x \\
+ k(y), \min_i \{ V(x, i + 1) + K(i, i + 1) - V(x, i), V(x, i) - V(x, i + 1) \} \right\} = 0,$$

subject to the condition $V(0, y) = 0$ and the requirement that $\lim_{x \to 1} V_x(x, y)$ exists and is finite.
Our analysis in the preceding section motivates us to consider the following functions:

\[
\psi_i(x, c) := e^{2\beta(1-x)/\sigma^2(1 - x) - 2\gamma/\sigma^2} \left[ \frac{2}{\sigma^2} \int_0^{x-1} e^{-2\beta u/\sigma^2} u^{2\gamma/\sigma^2 - 1} \left( \ell + \frac{K_i}{1 - u} \right) du + c \right]
\]  

for \(i = 1, \ldots, m\), plus the function \(\phi\) as defined by (12). In particular, note that, for each fixed \(c\), \(\psi_i(\cdot, c)\) is a solution function to the following first-order differential equation:

\[
U'_i(x) + \frac{2}{\sigma^2} \left( \beta_i - \frac{\gamma}{1 - x} \right) U_i(x) = -\frac{2}{\sigma^2} \left( \ell (x) + \frac{K_i}{x(1-x)} \right).
\]

Recall from the base-case scenario (i.e., \(m = 1\)) that in order for a sequential switching policy to be optimal, the fixed cost cannot be too high. Precisely, using the current set of notations, we would need to ensure that the area of intersection between \(\phi(\cdot, i)\) and \(\psi_1(\cdot, 0)\) should be greater than or equal to \(K_{0,1}\), so that some \(i^*\) exists to make the area of intersection between \(\phi(\cdot, i^*)\) and \(\psi_1(\cdot, 0)\) equal exactly \(K_{0,1}\). This motivates the following condition as a natural extension of Assumption 1 to ensure that a sequential switching policy utilizing all possible lockdown levels is optimal.

**Assumption 2** There exist \(i^* \leq \bar{i}\) and \((c_1^*, \ldots, c_m^*)\) such that the following conditions hold:

(i) \(\phi(\cdot, i^*)\) and \(\psi_1(\cdot, c_1^*)\) intersect twice at \(x_{1,0}\) and \(x_{0,1}\) and

\[
\int_{x_{1,0}}^{x_{0,1}} [\phi(x, i^*) - \psi_1(x, c_1^*)] dx = K_{0,1};
\]

(ii) for \(i = 1, \ldots, m - 1\), \(\psi_i(\cdot, c_i^*)\) and \(\psi_{i+1}(\cdot, c_{i+1}^*)\) intersect twice at \(x_{i+1,i}\) and \(x_{i,i+1}\), and

\[
\int_{x_{i+1,i}}^{x_{i+1,i}} [\psi_i(x, c_i^*) - \psi_{i+1}(x, c_{i+1}^*)] dx = K_{i,i+1};
\]

(iii) \(\lim_{x \to 1} \psi_m(x, c_m^*)\) is finite.

**Remark 2** It is easily verifiable from the explicit expressions given by (23) that part (iii) of Assumption 2 necessarily implies that \(c_m^* = 0\).

**Remark 3** Part (ii) of Assumption 2 implies that \(\psi_{m-1}(x, c_{m-1}^*) < \psi_m(x, c_m^*)\) for all \(x > x_{m-1,m}\). This observation, in turn, implies that \(c_{m-1}^* \leq 0\), because we have \(\lim_{x \to 1} \psi_m(x, c_m^*) < \infty\) but \(\lim_{x \to 1} \psi_{m-1}(x, c) = \infty\) for all \(c > 0\). Using backward induction, we can deduce that \(c_i \leq 0\) for all \(i = 1, \ldots, m - 1\). By the same token, we can conclude from part (i) of Assumption 2 that \(i^* \leq \bar{i}\). This is because \(\lim_{x \to 1} \psi_1(x, c_1^*) < \infty\) whereas \(\lim_{x \to 1} \phi(x, \bar{i}) = \infty\) for all \(i > \bar{i}\).
Theorem 2 (Sequential switching under multiple lockdown levels) If Assumption 2 holds with \( x_{1,0} < x_{2,1} < \cdots < x_{m,m-1} \) and \( x_{0,1} < x_{1,2} < \cdots < x_{m-1,m} \), then a sequential switching policy characterized by the two strongly ordered sequences is optimal for the problem described by (21). That is, the social planner switches from level \( i \) to \( i+1 \) \((i + 1 \text{ to } i)\) as soon as the infection level becomes above \( x_{i,i+1} \) (below \( x_{i+1,i} \)).

Proof of Theorem 2. The key to the proof is to construct a solution to (22). For this purpose, let \( V(x, y) \) be such that (a) \( V(0, 0) = 0 \), (b) \( V(x^{*}_{i+1,i}, i) = V(x^{*}_{i}, i+1) \) for \( i = 0, \ldots, m - 1 \), and (c)

\[
V_{x}(x, y) = \begin{cases} 
\phi(x, t^{*})1_{\{x<x_{1,0}^{*}\}} + \sum_{j=1}^{m} \psi(x, c_{j}^{*})1_{\{x_{j-1,j}^{*}<x<x_{j,j+1}^{*}\}} & \text{if } y = 0, \\
\phi(x, t^{*})1_{\{x<x_{1,0}^{*}\}} + \sum_{j=1}^{i-1} \psi(x, c_{j}^{*})1_{\{x_{j-1,j}^{*}<x<x_{j,j+1}^{*}\}} + \psi(x, c_{i}^{*})1_{\{x_{i-1,i}^{*}<x<x_{i,i+1}^{*}\}} \\
+ \sum_{j=i+1}^{m} \psi(x, c_{j}^{*})1_{\{x_{j-1,j}^{*}<x<x_{j,j+1}^{*}\}} & \text{if } y \neq 0.
\end{cases}
\]

By construction and applying Assumption 2, it is easy to see that \( V(x, y) \) satisfies (22). Given this, the remaining steps towards the desired conclusion will mimic those in the proof of Theorem 1. Hence, we omit the details.

We wish to point out that a violation of Assumption 2 does not necessarily mean that switching between different modes is not worthwhile. For example, it could be that all the conditions in Assumption 2 hold, but with \( m \) therein replaced by some \( j(< m) \) but not with \( j + 1 \). We interpret this case to mean that the social planner should consider adopting lockdown levels up to \( j \) but not to \( j + 1 \). This observation motivates us to devise an iterative scheme to compute the optimal intervention strategy with \( m + 1 \) modes. The idea is to examine one lockdown level at a time, starting from the lowest (i.e., level 1) to see if it is worth consideration, until a lockdown level is found to be not worth pursuing.

An iterative procedure. Suppose level \( l \) lockdown is considered worthwhile with \( l < m \). This would imply that (i) there exist \( t^{(l)} \) and \( (c_{1}^{(l)}, \ldots, c_{l}^{(l)}) \) that collectively satisfy a modified version of Assumption 2 (changing \( m \) therein to \( l \)), and (ii) the corresponding points of intersection are such that \( x_{1,0}^{(l)} < x_{2,1}^{(l)} < \cdots < x_{l,l-1}^{(l)} \) and \( x_{0,1}^{(l)} < x_{1,2}^{(l)} < \cdots < x_{l-1,l}^{(l)} \). To examine if level \( l + 1 \) lockdown is worthwhile, one computes the area of intersection between \( \psi_{l}(\cdot, c_{l}^{(l)}) \) and \( \psi_{l+1}(\cdot, 0) \), denoted as \( K_{l,l+1}^{(l)} \). \( K_{l,l+1}^{(l)} = 0 \) if the two functions graphs touch or do not intersect.
• If $K_{l,l+1}^{(l)} \leq K_{l,l+1}$, then level $l + 1$ lockdown is not worth consideration. Hence, the procedure terminates and concludes that a sequential switching policy with lockdowns up to level $l$ is optimal. In particular, the switching policy is characterized by two strongly ordered sequences: $x_{1,0}^{(l)} < x_{2,1}^{(l)} < \cdots < x_{l,l-1}^{(l)}$ and $x_{0,1}^{(l)} < x_{1,2}^{(l)} < \cdots < x_{l-1,l}^{(l)}$.

• If $K_{l,l+1}^{(l)} > K_{l,l+1}$, then one looks for $\iota^{(l+1)}$ and $(c_1^{(l+1)}, \ldots, c_{l+1}^{(l+1)})$ that collectively satisfy a modified version of Assumption 2 (changing $m$ therein to $l + 1$).
  - If the points of intersection are such that $x_{1,0}^{(l+1)} < x_{2,1}^{(l+1)} < \cdots < x_{l+1,l}^{(l+1)}$ and $x_{0,1}^{(l+1)} < x_{1,2}^{(l+1)} < \cdots < x_{l,l+1}^{(l+1)}$, then level $l + 1$ lockdown is considered worthwhile, in which case the procedure continues by setting $l \leftarrow l + 1$ (unless $l + 1 = m$);
  - otherwise, the procedure stops and concludes the previously identified switching policy with lockdowns up to level $l$ is optimal.

Finally, we consider a numerical example having three intervention levels (with $m = 2$) to illustrate our results in Theorem 2. Let $\beta = 0.45$, $\beta_1 = 0.2$, $\beta_2 = 0.1$, $\gamma = 1$, $\sigma = 0.5$, $\ell = 6$, $\kappa_1 = 0.4$, $\kappa_2 = 0.6$, and $K_{0,1} = 0.5$, $K_{1,2} = 0.45$. Using the iterative scheme described previously, we compute $\iota^* = 27.42$, $c_1^* = -0.0015$, $c_2^* = 0$, $x_{1,0} = 0.014$, $x_{0,1} = 0.301$, $x_{2,1} = 0.030$, $x_{1,2} = 0.778$. In the left-hand panel of Figure 3 we graph three functions: $\phi(x, \iota^*)$, $\psi_1(x, c_1^*)$ and $\psi(x, c_2^*)$ as functions of $x$. Here, the areas of the two shaded regions are equal to $K_{0,1}$ and $K_{1,2}$. Because $x_{0,1} < x_{1,2}$ and $x_{1,0} < x_{2,1}$, we conclude that the optimal sequential switching policy as described in Theorem 2 is optimal. On the other hand, when we increase $\kappa_2$ from 0.6 to 0.68 (so that level-2 lockdown becomes more costly, the optimal policy stipulates that we will never activate the level-2 lockdown because the bottom shaded area is smaller than $K_{0,1}$ as shown in the right-hand panel of Figure 3.)
Figure 3: Illustration of the optimal switching policy with multiple lockdown levels: $m = 2$, $\beta = 0.45$, $\beta_1 = 0.2$, $\beta_2 = 0.1$, $\gamma = 1$, $\sigma = 0.5$, $\ell = 6$, $K_{0,1} = 0.5$, $K_{1,2} = 0.45$, $\kappa_1 = 0.4$, $\kappa_2 = 0.6$ (left) and 0.68 (right).

6 Concluding Remarks

The emergence of novel human pathogens, such as COVID-19, and the devastating health and socioeconomic consequences for societies have highlighted the need for the development of decision-support tools to guide unconventional government interventions that do not rely on vaccines. In contrast to most existing results considering pharmaceutical interference, we proposed the use of an optimal control framework to characterize and identify the best timing for governments to step in and step out. We showed that whether the government should intervene and when to react are greatly affected by both the biological characteristics of the infectious disease and the socioeconomic costs associated with the intervention.

The model considered in this paper, albeit capturing usefully some key realistic features of an infectious disease, has ignored the small but nonzero disease incubation period. The incorporation of a positive incubation period will complicate the analysis in the following ways: (i) one would need to track the number of individuals who have been exposed yet have not developed symptoms; and (ii) the number of new infections may not be directly observable to the social planner. Thus, relaxing such an assumption would not only enlarge the state space but
also give rise to a control problem with partial observation. Furthermore, it is noteworthy that
the specific types of costs we have considered thus far, while reasonable, are motivated as much
by mathematical convenience as by realism. Both the cost of disease and the cost of intervention
may take on more complicated forms, making it far more difficult to extract structural insights
from model output.

References


Mathematics and Computation 305:221–240.


Chen Z, Kong G (2022) Hospital admission and social distancing: An SEIR model with constrained


Probability 239–260.

El Housni O, Sumida M, Rusmevichientong P, Topaloglu H, Ziya S (2022a) Can testing ease social

El Housni O, Sumida M, Rusmevichientong P, Topaloglu H, Ziya S (2022b) Future evolution of


